

VISUALIZATION OF ELECTROMAGNETIC FIELDS USING AWK

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The solution of boundary-value problems is of central importance to nearly every field of physics. The first introduction to this topic for most university physics students is in the classical electromagnetism courses. In these classes, students are exposed to basic techniques such as the method of images, variable separation, and the development of solutions to the Laplace equation by series. These methods are limited in either application or the ease with which basic data such as charge densities on conductors can be extracted from the solutions.

The most powerful method for solving the Laplace equation in two-dimensional applications is the only method that fully exploits the extraordinary level of symmetry of the electromagnetic field: conformal mapping. Recall that the Laplace equation in two dimensions can be written in complex form as

$$\nabla^2 \Phi = 0 = \frac{\partial^2}{\partial z \partial \bar{z}} \Phi,$$

using $z = x + iy$, $\bar{z} = x - iy$. This means that any function of z alone,

$$\Phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n,$$

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solves the Laplace equation. We simply choose the function to satisfy the boundary conditions. It is particularly easy to extract information such as field components and charge densities on conductors from this form of the potential. In any region in which the potential is analytic, we can write $\Phi(z) = U(x,y) + iV(x,y)$, in which U and V satisfy the Cauchy-Riemann conditions. It is trivial to show that

$$E_x + iE_y = \left(\frac{d\Phi}{dz} \right)^*$$

and that on any conducting surface S ,

$$|\sigma| = \epsilon_0 \left| \frac{d\Phi}{dz} \right|_S$$

is the charge density on S .

Suppose we perform a variable change

$$w = w(z), \quad \frac{\partial}{\partial z} = \frac{dw}{dz} \frac{\partial}{\partial w}.$$

Then

$$\nabla^2 \Phi(z) = \left| \frac{dw}{dz} \right|^2 \frac{\partial^2}{\partial w \partial \bar{w}} \Phi = 0,$$

and so as long as the derivative exists in the region of interest,

$$\frac{\partial^2}{\partial w \partial \bar{w}} \Phi(w) = 0$$

and the Laplace equation is unchanged in the new coordinate system $w = u + iv$.

The change of variables $(x, y) \rightarrow (u, v)$ preserves the angle of intersections of families of curves; two curves in the x - y plane that are normal to one another are mapped to curves normal to one another in the u - v plane. If we solve some trivial problems in the x - y plane, we can map them to nontrivial problems in the u - v plane.

The method of conformal transformation is often neglected in university electromagnetism courses due to the mathematical expertise that is prerequisite to its use. However, the technique is rewarding for several reasons. Not only can it essentially sum the standard series solutions found by the methods of separation of variables, resulting in closed forms for the voltage function, but also the extraction of information from the solutions it gives is particularly easy. The technique hints at the deepest underlying symmetry of the electromagnetic field, which has served as a model for some of the most successful theories of modern physics. It also provides the most powerful framework in which to generate visualizations of the field geometry. This last divi-

Software Availability

Gnu Awk can be obtained from the Free Software Foundation by FTP at <ftp://prep.ai.mit.edu/pub/gnu/gawk-3.0.3.tar.gz>.

The graphing tools mentioned in the article are also available from the same source: <ftp://prep.ai.mit.edu/pub/gnu/gnuplot-3.5.tar.gz> or <ftp://prep.ai.mit.edu/pub/gnu/plotutils-1.3.tar.gz>.

dend provided the motivation for developing a simple computational method for performing the conformal mappings in a classroom setting.

The visualization process outlined in this article is simple. We will generate points on the equipotential and field-line curves for trivial, exactly soluble systems such as infinite parallel-plate conductors and infinite line charges of uniform charge density. The points will be stored in a two-column, x - y format. We will then illustrate how the line-processor program Awk can be used to perform pointwise conformal transformations on these data files, in order to generate the points on the equipotentials and field lines of virtually any conductor or line-charge arrangement in which the voltage depends on only two variables. In keeping with the notion in electrostatics that every complex potential solves two boundary-value problems, by simply exchanging the roles of the real and imaginary parts as field lines and equipotentials, we illustrate two points. Conformal mapping is a potent tool in electromagnetism, and Awk can be used in ways perhaps not predicted by its authors as a powerful visualization tool in its own right.

Computer-aided conformal mapping

In order to generate the equipotentials and field lines of any conductor or charge arrangement, it is sufficient to begin with those of parallel plates. We write the solution to the Laplace equation as $\Phi = \text{Im } z$, and the equipotentials and field-line curves are respectively $c = \text{Re } z$ and $c' = \text{Im } z$, since these two families of curves are mutually orthogonal. By conformal mapping to a new complex variable $z \rightarrow w = f(z)$, we generate solutions to the Laplace equation in the $w = u + iv$ plane with equipotentials and field lines given by

$$c = \text{Re } f^{-1}(w), \quad c' = \text{Im } f^{-1}(w),$$

respectively. All these transformations can be done point by point on the data file containing the points for the equipotentials and field lines of the parallel-plate capacitor using simple Awk scripts.

Awk is a powerful programming language available for nearly every platform (see "Software Availability," this page) and used to reformat data line-by-line. Awk scripts can be executed from the shell or as program files. For example, suppose that we have a data file consisting of hundreds of lines, each with two floating-point numbers per line, separated by a space. In this case the character Awk recognizes as a data-field separator is the blank space, and the record

separator is the new-line character. From the command line of the shell we use Awk to print out the geometric mean of the two numbers on each line with

```
awk '{print $1," ",$2," ",sqrt($1*$2)}' data
```

or we could write the script to a file and filter the data through it with

```
#Mean.awk
{
x=$1
y=$2
z=sqrt(x*y)
print x," ",y," ",z
}
```

with the command `awk -f Mean.awk data`. The output is to the console, but it can be piped to another script or directed into a file.

One of the most potentially useful techniques for solving boundary-value problems in which charges are present is the method of images. However, the scope of the method is limited unless it is used in conjunction with conformal mapping to alter the location and shape of bounding conductors.

For example, consider a line charge λ a distance of one unit above a grounded plane. We know from the method of images that

$$\begin{aligned} \Phi &= \frac{\lambda}{2\pi\epsilon_0} \ln \sqrt{\frac{x^2 + (y-1)^2}{x^2 + (y+1)^2}} \\ &= \text{Re} \frac{\lambda}{2\pi\epsilon_0} \ln \frac{z-i}{z+i}. \end{aligned}$$

Field lines and equipotentials are orthogonal families of circles whose analytical forms are trivial to find. The first step in generating literally dozens of solutions to boundary-value problems from this single example is to create a data file containing the points on the equipotentials and field lines. The C++ program below does this for us.

```
#include<iostream.h>
#include<math.h>

/* Draws field lines for line charge
above a grounded plane (images) */

main ()
{
double dc = 0.1;
double dtheta = 0.01;
for (int n = 1; n <= 25; n++)
{
double c = double (n) * dc;
double u = 1.0 / tan (c);
for (int m = 1; m <= 157; m++)
{
double theta = double (m) * dtheta;
double r = u * cos (theta) +
sqrt (u * u * cos (theta) * cos (theta) + 1.0);
```

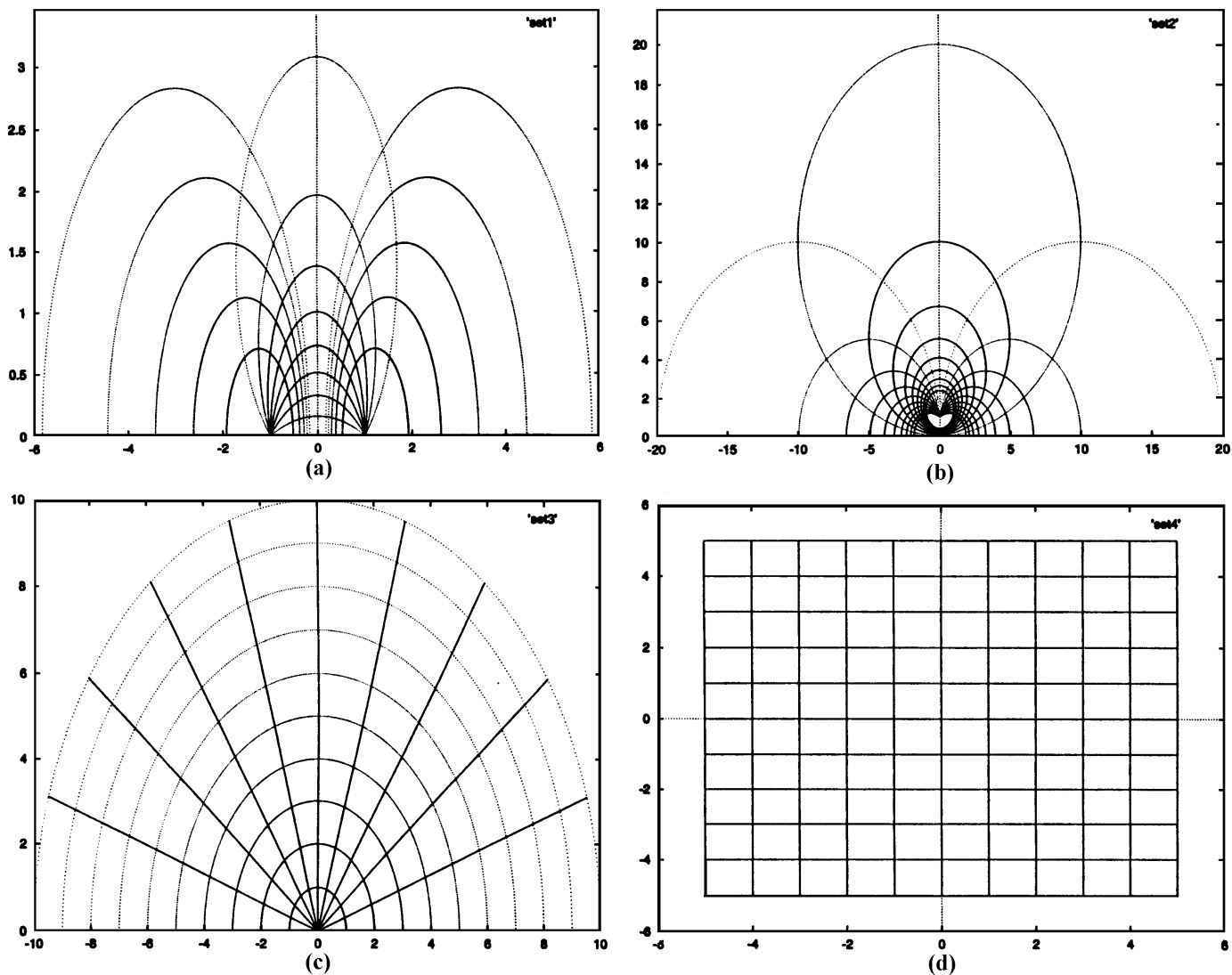


Figure 1. Equipotentials and field lines in four exactly soluble sets of field geometries: (a) grounded plane with a high-voltage strip from $x = -1$ to $x = 1$ (set 1); (b) line charge above a grounded plane (set 2); (c) line charge through the origin (set 3); and (d) parallel plates (set 4).

```
    cout << r * cos (theta) << " " << r * sin (theta) << "\n";
  }
}
for (int n = 1; n <= 25; n++)
{
  double c = double (n) * dc;
  double u = -1.0 / tan (c);
  for (int m = 1; m <= 157; m++)
  {
    double theta = 3.1415 - double (m) * dtheta;
    double r = u * cos (theta) +
      sqrt (u * u * cos (theta) * cos (theta) + 1.0);
    cout << r * cos (theta) << " " << r * sin (theta) << "\n";
  }
}
return (0);
}
```

To compile the programs to produce the data for field lines and equipotentials, and dump the data into a pair of files, we execute the commands

```
g++ field_lines1.cc
./a.out>field_lines1
g++ equipotentials1.cc
./a.out>equipots1
cat equipots1 field_lines1>set2
```

The file set2 contains the points for both sets, which we can graph with any plotting routine such as SigmaPlot, gnuplot, or plotutils. On our Linux computers using the Gnu plotutils graphing routines, we would use the command `graph-X -m 0 set2` to preview the graph in X-Windows and the command `graph -m 0 set2 | plot2ps>set2.ps` to create a printable PostScript file. Alternatively we can create a much smaller PostScript file with gnuplot by means of the sequence of commands

```
set terminal postscript
set output 'set2.ps'
plot 'set2' with dots'
```

The result of this sequence is shown in Fig. 1b, in which the equipotentials are the closed curves.

By starting with the field lines and equipotentials for two or three cases such as an infinite parallel-plate capacitor, a line charge through the origin, and perhaps the configuration just considered, we can solve literally hundreds of boundary-value problems. We solve these problems by filtering the points on the two families of curves through 20 Awk scripts that perform special conformal transformations (see table).

Most of these scripts will be obvious, such as the following:

```
#map2.awk
# w=ln(z)
{
x=$1
y=$2
u=log(sqrt(x*x+y*y))
v=atan2(y,x)
print u," ",v
}
```

which performs $w = \ln z$. The eleventh mapping is less obvious. It computes the real and imaginary parts of an elliptic integral of a complex angle using the well-known formula

Table. Awk scripts used in conformal mapping.			
Label	Transform	Label	Transform
map1.awk	$w = \sin^{-1} z$	map11.awk	$w = F(\phi + i\psi)$
map2.awk	$w = \ln z$	map12.awk	$w = \sin \frac{\pi z}{a}$
map3.awk	$w = i \frac{1-z}{1+z}$	map13.awk	$w = z + a$
map4.awk	$z = \frac{1+iw}{1-iw}$	map14.awk	$w = \sqrt{z}$
map5.awk	$w = z + \frac{1}{z}$	map15.awk	$w = \sin^{-1} \frac{z}{c}$
map6.awk	$w = z^{1/4}$	map16.awk	$w = z + ia$
map7.awk	$w = z + \sqrt{z^2 - 1}$	map17.awk	$w = \frac{z-1}{z+1}$
map8.awk	$w = z - \frac{1}{z}$	map18.awk	$w = i \frac{1+z}{1-z}$
map9.awk	$w = z + \sqrt{z^2 + 1}$	map19.awk	$w = e^{i\theta} z$
map10.awk	$w = e^{az}$	map20.awk	$w = \frac{1}{z}$

$$F(\phi + i\psi/m) = F(\lambda/m) + iF(\mu/|1-m|),$$

where $\cot^2 \lambda$ is the positive root of the quadratic

$$x^2 - [\cot^2 \phi + m \sinh^2 \psi \csc^2 \phi - 1 + m]x - (1-m)\cot^2 \phi = 0$$

and

$$m \tan^2 \mu = \tan^2 \phi \cot^2 \lambda - 1.$$

This mapping calls on user-defined functions in the Awk script as well as a loop to compute the elliptic integral by a Landen series. The third through the fifth lines in the script illustrate how a user-defined function is created. This particular script has three such functions: $\sinh(x)$, $\operatorname{asin}(x)$, and $F(\phi, m)$.

```
#map11.awk
# Compute elliptic integral of complex angle phi+i*psi

function sinh(x) {
return (exp(x)-exp(-x))/2.0
}

function asin(x) {
return atan2(x,sqrt(1.0-x*x))
}

function F( phi, m) {
sine[0]=sqrt(m)
cose[0]=sqrt(1.0-m)
Phi[0]=phi
prod=2.0/(1.0+sine[0])
for(i=1;i<=10;++i) {
cose[i]=(1.0-sine[i-1])/(1.0+sine[i-1])
sine[i]=sqrt(1.0-cose[i]*cose[i])
Phi[i]=0.5*(Phi[i-1]+asin(sine[i-1]*sin(Phi[i-1])))
prod=prod*2.0/(1.0+sine[i])
}

return
prod*log(sin(0.785398+Phi[10]/2.0)/cos(0.785398+Phi[10]/2.0))
}

#main routine
{
phi=$1
psi=$2
m=0.5
a=cos(phi)
b=sin(phi)
c=sinh(psi)
factor=a*a+m*c*c-(1.0-m)*b*b
rad=factor+sqrt(factor*factor+4.0*(1.0-m)*a*a*b*b)
lambda=atan2(sqrt(2.0*b*b), sqrt(rad))
new=rad-2.0*a*a
if(new<0.0){
new=-new}
mu=atan2(sqrt(new),sqrt(2.0*m*a*a))
```

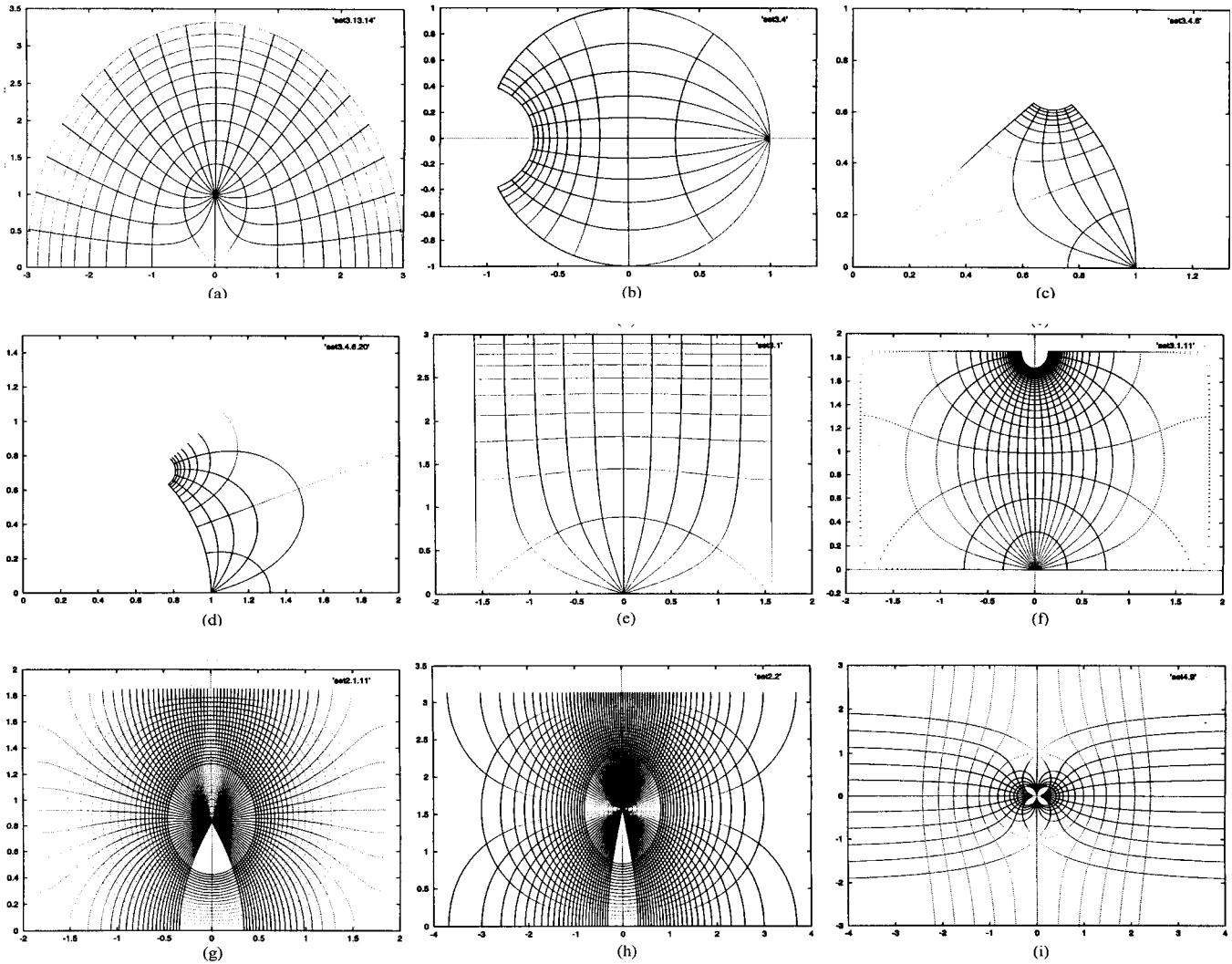


Figure 2. Equipotentials and field lines for various field geometries: (a) line charge above an insulated plane; (b) cylinder with upper half grounded and lower half at V_0 ; (c) wedge with boundaries at different potentials; (d) external fields of a wedge with boundaries at different potentials; (e) conducting channel with walls at different potentials; (f) rectangular box with halves at different potentials; (g) line charge in a grounded conducting box; (h) line charge between two grounded parallel plates; and (i) grounded conducting cylinder in a uniform electric field.

```
printF(lambda,m),"",F(mu,1.0-m)
}
```

Awk supports a number of basic programming features such as conditionals, loops, user-defined functions, and variable declarations, and the syntax is particularly simple. In addition, Awk contains several predefined mathematical functions performed in double precision, such as \sin , \cos , atan2 , sqrt , \log , and exp . A parameter can be passed to the script at run time; for example, `awk -f map16.awk a=2.0 data` will perform $z \rightarrow w = z + 2.0i$. The parameter a is an undeclared variable that appears in the script without being assigned a value.

As you can see from the example of `map11.awk`, any programming experience in any language will allow you to understand the basic features of Awk and to write Awk scripts. The amount of syntax is small, and most of it is of an obvious nature.

The beauty of this approach to constructing electrostatic fields lies in the ability of the shell to pipe the output of one program into another. We can perform consecutive conformal mappings of our field lines and equipotentials in an effortless way; for example, `awk -f map1.awk data | awk -f map2.awk | awk -f map3.awk >newdata` will perform three sequential mappings on the same data file, resulting in the file `newdata`.

Some pretty results

The main drawback to the conventional series solutions to the Laplace equations is the fact that they are series. It may be extremely difficult to extract the equations of field lines and equipotentials from the series solution for the potential.

Using Awk and conformal mapping, we can now illustrate the field geometries for the classic electrostatic examples, as well as some cases that are far too difficult to tackle by other methods. These classic examples and difficult cases

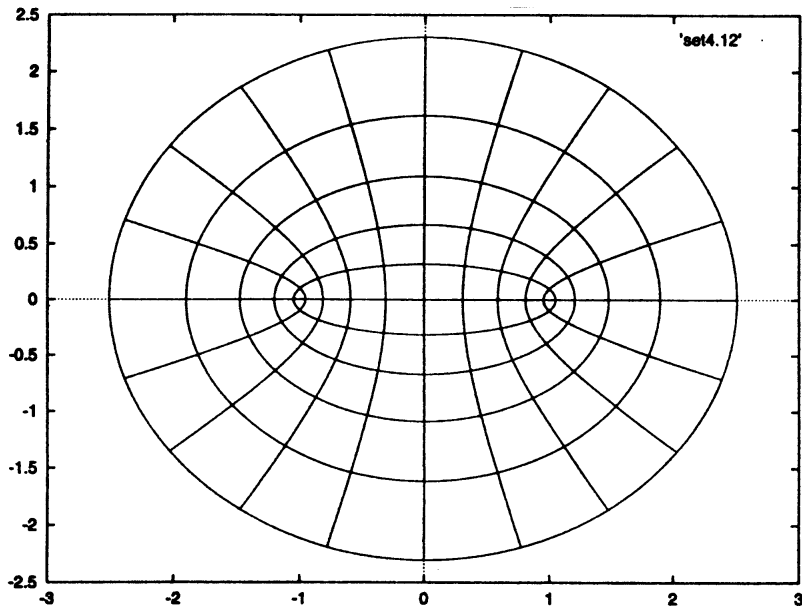


Figure 3. Equipotentials and field lines surrounding a conducting narrow strip.

will be presented below. First, however, we consider four elementary, exactly soluble sets of field geometries: (a) a grounded plane with a high-voltage strip from $x = -1$ to $x = 1$, which we call set 1; (b) a line charge above a grounded plane (discussed above), which we call set 2; (c) a line charge through the origin, which we call set 3; and (d) parallel plates, which we call set 4. The equipotentials and field lines for these cases have been collected in Fig. 1. If we exchange the role of equipotential and field lines in set 3 (see Fig. 1c), these curves also describe a grounded half-plane joined to a high-voltage half-plane along an axis perpendicular to the page.

Line charge above an insulated plane. The solution to the problem of a line charge above an insulated plane can be solved by images:

$$\begin{aligned} \Phi(x,y) &= \frac{\lambda}{2\pi\epsilon_0} \ln \sqrt{[(x^2 + (y-1)^2)][x^2 + (y+1)^2]} \\ &= \text{Re} \frac{\lambda}{2\pi\epsilon_0} \ln (z-i)(z+i). \end{aligned}$$

The field lines and equipotentials can be gotten from those of a line charge through the origin by the inverse map of $z \rightarrow w = (z-i)(z+i)$, namely $z = \sqrt{w-1}$ or awk -f map13.awk a=1.0 set3 | awk -f map14.awk >set3.13.14, which after a 90° rotation results in the curves shown in Fig. 2a.

Cylinder with upper half grounded and lower half at V_0 . For our next sequence, regard set 3 as the fields for two conducting half-planes joined along the z -axis. The entire upper-half of space can be mapped into the interior of a cylinder with

$$w = \frac{1+iz}{1-iz}.$$

For the two half-planes, the equipotentials and field lines are respectively

$$C = \frac{V_0}{\pi} \text{Im} \ln z, \quad C' = \frac{V_0}{\pi} \text{Re} \ln z.$$

For the region within a cylinder with the upper half grounded and the lower half at voltage V_0 these become

$$C = \frac{V_0}{\pi} \text{Im} \ln \frac{i(1-w)}{1+w}, \quad C' = \frac{V_0}{\pi} \text{Re} \ln \frac{i(1-w)}{1+w},$$

and the field geometry is drawn in the u - v plane by awk -f map4.awk set3>set3.4. The equipotentials and field lines are shown in Fig. 2b.

Wedge with boundaries at different potentials. Since in Fig. 2b the $\text{Re } w = u$ axis is at voltage $V_0/2$, we can solve the problem of finding the fields within a 45° wedge with outer curved boundary grounded and sides at $V_0/2$ with awk -f map6.awk set3.4>set3.4.6. The conformal mapping $w = z^{1/4}$ will create a bend in the real axis at the origin with a 45° angle and send the entire upper half of the z -plane into the interior of the wedge. Results are in Fig. 2c.

External fields for a wedge with boundaries at different potentials. The exterior problem ($r \geq 1$) can be solved by filtering the data for the last set of curves with awk -f map20.awk set3.4.6>set3.4.6.20, utilizing the map $w = 1/z$, which sends the interior of the unit circle into the entire region outside of it. The results are shown in Fig. 2d.

In this case we find a potential function (up to an additive constant)

$$\begin{aligned} \Phi &= \frac{V_0}{\pi} \text{Im} \ln z \rightarrow \frac{V_0}{\pi} \text{Im} \ln \frac{i(1-w)}{1+w} \rightarrow \frac{V_0}{\pi} \text{Im} \ln \frac{i(1-w^4)}{1+w^4} \\ &\rightarrow \frac{V_0}{\pi} \text{Im} \ln \frac{i(1-w'^{-4})}{1+w'^{-4}} \\ &= \frac{-2V_0}{\pi} \text{Im} (r^{-4} e^{-4i\theta} + \frac{1}{3} r^{-12} e^{-12i\theta} + \frac{1}{5} r^{-20} e^{-20i\theta} + \dots) \\ &= \frac{2V_0}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} r^{-4(2n+1)} \sin(4(2n+1)\theta), \end{aligned}$$

using the polar form of $w'' = u'' + iv''$ after all of the transformations have been done. This is of course the solution to one of the standard undergraduate electromagnetism problems: the potential within a conducting wedge with grounded walls and inner conducting surface at $r = a = 1$ held fixed at V_0 .



Conducting channel with walls at different potentials.

The problem of finding the field geometry in a conducting channel with infinitely high walls, left side at V_0 and right side grounded, is solved by `awk -f map1.awk set3>set3.1`. The transformation $z = \sin w$ is precisely what the Schwartz-Christofel transformation says that we need in order to put two 90° bends in the real axis at $z = \pm 1$:

$$w = \int^z \frac{dz'}{\sqrt{1-z'}\sqrt{1+z'}} = \sin^{-1} w .$$

In the w -plane the equipotentials and field lines are as in Fig. 2e.

The actual voltage function can be gotten in closed form as

$$\Phi = \frac{V_0}{\pi} \text{Im} \ln z \rightarrow \frac{V_0}{\pi} \text{Im} \ln \sin w = \frac{V_0}{\pi} \tan^{-1} \frac{\cos u \sinh v}{\sin u \cosh v} ,$$

with $w = u + iv$.

Rectangular box with halves at different potentials. We can now solve a nontrivial problem: find the field geometry within a rectangular box, whose left half is at high voltage and right half is at ground, by putting four bends, each of 90° , in the real axis and mapping the entire upper half z -plane into the interior of a rectangle with

$$w = \int^z \frac{dz'}{\sqrt{1-z'}\sqrt{1+z'}\sqrt{1-kz'}\sqrt{1+kz'}} = F(z|m) ,$$

an elliptic integral of modulus $m = k^2$. The inverse of this function is the Jacobi elliptic function $z = \text{sn}(w | m)$, and so the potential becomes

$$\Phi = \frac{V_0}{\pi} \text{Im} \ln \text{sn}(w|m) .$$

Setting the voltage constant, we obtain the equations of equipotentials. We can map the equipotentials and field lines of the line charge into the curves for this problem with `awk -f map1.awk set1 | awk -f map11.awk > set3.1.11`, since our script for the elliptic integral requires that z be expressed as the sine of a complex angle. The results are in Fig. 2f.

Line charge in a grounded conducting box. If we begin with the fields of a line charge above a grounded conducting plane, we can construct the fields for a line charge within a grounded conducting rectangular box by using the commands `awk -f map1.awk set2 | awk -f map11.awk > set2.1.11`. The results are in Fig. 2g.

Line charge between two grounded parallel plates. We construct the fields for a line charge in the region between two grounded parallel plates by using the transformation $w = \ln z$, since the positive real axis will be mapped to the entire real axis in w -space and the negative real axis will be mapped to the line $\ln(-x) = \ln|x| + i\pi$. The transformation is performed by `awk -f map2.awk set2>set2.2`, resulting in the potential

$$\Phi = \frac{\lambda}{2\pi\epsilon_0} \text{Re} \ln \frac{e^w - i}{e^w + i} ,$$

shown in the w -plane in Fig. 2h.

Grounded conducting cylinder in a uniform electric field.

The classic undergraduate boundary-value problem of a grounded conducting cylinder in an otherwise uniform electric field has the solution

$$\Phi(r, \theta) = -E_0 \left(r - \frac{1}{r}\right) \cos \theta = \text{Re} \left[-E_0 \left(z - \frac{1}{z}\right)\right] .$$

This solution can be obtained from the solution for a uniform field by `awk -f map9.awk set4>set4.9`, in which the grounded cylinder is the circle of unit radius in Fig. 2i.

Conducting narrow strip. As a final example, we construct the fields surrounding a conducting narrow strip from the uniform field by `awk -f map12.awk a=10.0 set4>set4.10`. This example beautifully illustrates the nonuniform charge distribution on a conductor. The complex voltage function is $\Phi(z) = V_0 z$ for the parallel-plate fields, and for the strip it is

$$\Phi(w) = \Phi(u, v) = \frac{10V_0}{\pi} \sin^{-1} w .$$

The charge density on the strip $-1 \leq u \leq 1$ is

$$|\sigma| = \epsilon_0 \left| \frac{d\Phi}{dw} \right|_{w=u} = \frac{10V_0\epsilon_0}{\pi} \frac{1}{\sqrt{1-u^2}} .$$

For the equipotentials and field lines, see Fig. 3.

Magnetostatics

The method of conformal mapping is not restricted to electrostatics. Consider magnetic fields made by steady currents (call them J_\perp) in the Coulomb gauge that are always perpendicular to the x - y plane. Then the vector potential has only one component satisfying $-\nabla^2 A_\perp = \mu_0 J_\perp$. In the empty space around the currents $-\nabla^2 A_\perp = 0$, and so $A_\perp(x, y) = A_\perp(z, \bar{z}) = A_\perp(z)$. We can bound regions of space with infinitely permeable boundaries ("iron") or superconducting boundaries and apply all the same methods as above. The only restriction is that only $C = \text{Re} A_\perp$ or $C' = \text{Im} A_\perp$ has physical significance for magnetic field lines.

Advantages

Awk can be effectively used to perform complex or difficult operations on data files. It is fast, easy to learn, and available for all platforms. Powerful mathematical concepts such as conformal symmetry of the solutions to the Laplace equation together with Awk can provide visualization of electrostatic and magnetostatic fields at a level and accessibility that would be difficult to rival with even the most advanced computational tools.