From Hamiltonian chaos to Maxwell's Demon

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The problem of the existence of Maxwell's Demon (MD) is formulated for systems with dynamical chaos. Property of stickiness of individual trajectories, anomalous distribution of the Poincaré recurrence time, and anomalous (non-Gaussian) transport for a typical system with Hamiltonian chaos results in a possibility to design a situation equivalent to the MD operation. A numerical example demonstrates a possibility to set without expenditure of work a thermodynamically non-equilibrium state between two contacted domains of the phase space lasting for an arbitrarily long time. This result offers a new view of the Hamiltonian chaos and its role in the foundation of statistical mechanics. © *1995 American Institute of Physics.*

I. INTRODUCTION

Maxwell's Demon (MD) was described by Maxwell in his book "Theory of Heat" (1871) and represented a rather simple, as it seemed, device. Two chambers, A and B, are separated by a division with a small hole in it (Fig. 1). Positioned at the hole the demon allows swifter molecules to pass from chamber A to B and slower molecules to pass from B to A. After a while chamber A will contain mainly swifter molecules, whereas chamber B— slower ones. This implies the appearance of temperature difference without expenditure of work and contradicts the Second Law of Thermodynamics.

Since the time it was invented, MD has experienced repeated shocks; the demon has been exorcized and summoned many times. The history of MD can be readily traced from a recent publication of a collection of the major articles devoted to MD in Ref. 1. The introduction written by the editors of this book gives a good outlook of the evolution of the MD concept and the contemporary views of MD. At present the concepts of MD's design utilize not only the laws of thermodynamics[2,3] but also some ideas of the information theory,[4,5] computer operations,[6,7] and the quantum nature of recognizing and handling the molecules that pass through the hole in the division.[8]

The purpose of this paper is to reconsider the idea of MD using contemporary concepts of dynamical chaos in the design of the Demon. At this point we would like to return to the year 1871 and to quote Maxwell himself: "In dealing with masses of matter, while we do not perceive the individual molecules, we are compelled to adopt what I have described as the statistical method of calculations, and to abandon the strict dynamical method, in which we follow every motion by the calculus."[9] We emphasize that to create a device operating within the laws of thermodynamics, one should (according to Maxwell) abandon the idea to trace the trajectories of individual particles, i.e. to abandon the use of purely dynamical laws.

Boltzmann's kinetic theory[10] should be considered the first successful attempt to unite in a formal way the statistical and dynamical description of a system. Boltzmann's contemporaries did not really understand that his idea was based essentially upon the existence of two different scales in time and space, the smaller one τ_c applicable to a purely dynamical process of collisions, and the larger one $\tau_d \gg \tau_c$ applicable to the statistical process of the system's evolution described by a kinetic equation. The resulting equilibrium distribution, provided it exists, corresponds to the thermodynamical state, although non-equilibrium states are also possible.

Boltzmann's ideas, his kinetic equation and H-theorem, were not appreciated by his contemporaries and were criticized, in particular, by Zermelo.[11] Only after Boltzmann's death, when a thorough statistical analysis of his theory had been made by Ehrenfest and Ehrenfest[12] and also the first numerical analysis had been performed ("urn model" of Ehrenfest), the validity of Boltzmann's ideas was acknowledged. The critical comments by Zermelo known as Zermelo's paradox were based on Poincaré Recurrence Theorem[13] and contained flawed statements which are worth discussing (see Section II and also Ref. 14).

In the following years many scientists were concerned with a problem of statistical physics foundation. This problem can be briefly formulated as follows: how (and in which form) to obtain the system's kinetic description based on the first principles, i.e. based on the system's Hamiltonian and possibly on some accurately formulated supplementary conditions. This has been successfully achieved in different ways and with varying degrees of generalization (see Refs. 15–18; there are, however, many other publications that are relevant). In these works the random phase approximation, the Gaussian nature of collision microprocesses, or equivalent conditions played the role of the statistical element which had to be added to the Hamiltonian dynamical equations to obtain a kinetic equation. The averaging over micro-random variables resulted in a reduced description of a system in the space of generalized actions or moments.

The theory of dynamical chaos has changed the view of the possibility of statistical laws foundation, since the dynamical trajectories, being the solutions of the deterministic equations of motion, may resemble the curves representing a random process. As he analyzed the foundations of quantum mechanics, Einstein made a well-known remark, "God doesn't play dice with the world."[19] Now we can add to those words that from the standpoint of classical physics only, God does not have to play dice, since dynamical equa-

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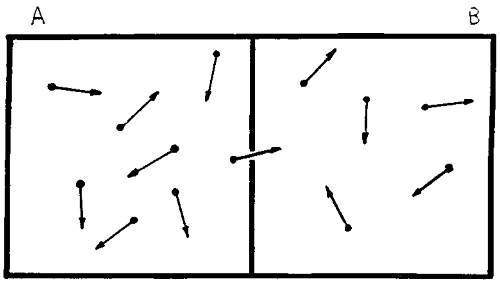

FIG. 1. A sketch of the original Maxwell's Demon design.

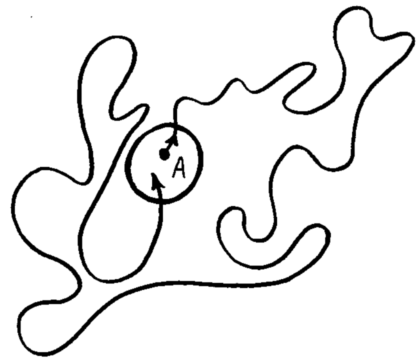

FIG. 2. Poincaré recurrence.

tions can themselves give rise to stochastic processes if certain simple restrictions for the parameters and initial conditions are applied. The use of the dynamical chaos phenomenon for statistical physics foundation started with Krylov's work[20] and continued for classical and quantum physics.[14,21,22] Strictly speaking the presence of dynamical chaos in a system is not sufficient for the foundation of the statistical physics laws, or, more accurately, this foundation has not been built so far. The major obstacles in substantiating statistical laws are how the dynamical chaos is structured in real Hamiltonian systems and what the real kinetics of such systems looks like. It was shown in Refs. 23–25 that kinetic description can be presented (at least over a sufficiently long time interval), as a fractional generalization of Fokker–Planck–Kolmogorov equation which does not possess the properties of a regular kinetic equation. The difference between the real Hamiltonian chaos and the conventional understanding of the laws of statistical physics can be demonstrated using the concept of MD, which is the main purpose of this paper. It will be demonstrated that MD can be realized in the conditions of dynamical chaos for, at least, an arbitrarily long period of time.

II. POINCARÉ RECURRENCES

According to Poincaré Recurrence Theorem a trajectory of a Hamiltonian with system finite motion, having started from some small area A, returns to this area the infinite number of times. Let T_j be a time interval between the $(j-1)$-th and j-th return. The variable T_j is an element of an infinite sequence:

$$\{T\} \equiv T_1, T_2, \ldots, \tag{2.1}$$

and will be referred to as a recurrence time.

In his critical comments[11] Zermelo assumed that recurrences occurred quasiperiodically which should have contradicted Boltzmann's H-theorem about the enthropy increase. Boltzmann rightly argued that for a large number of particles this time would be astronomically long. Unfortunately, Boltzmann's argument can not be used for the systems with dynamical chaos, since the latter is possible in the systems with as little as two interacting particles. For that reason we were compelled to make a more thorough analysis of the problem of Poincaré recurrences.

In fact, neither quasiperiodical recurrences nor any information at all about the nature of the $\{T\}$ sequence (2.1) follows from Poincaré Recurrence Theorem. In conditions of dynamical chaos the sequence (2.1) is random, and it is possible to raise the question of the probability distribution for recurrence time $f(T)$ which has to be normalized

$$\int_0^\infty f(T)\,dT = 1. \tag{2.2}$$

The properties of $f(T)$ has been studied for a number of models of dynamical chaos[26–29] and in "normal" conditions which will be explained below,

$$f(T) = (1/\langle T\rangle)\exp(-T/\langle T\rangle), \tag{2.3}$$

where $\langle T\rangle$ is an average recurrence time.[29] For the simplest cases one can show that

$$\langle T\rangle = 1/h, \tag{2.4}$$

where h is Kolmogorov–Sinai entropy (see the example in Ref. 29).

The results (2.3), (2.4) can be qualitatively explained in the following way. It is known for systems with dynamical chaos that the number of periodical orbits with period less then T is asymptotically equal[30] to the following:

$$N(T) \sim \exp hT \quad (T \to \infty). \tag{2.5}$$

If recurrence time is very large, a cycle of the unstable periodical curve differs only slightly from the random nonperiodic curve that returns to a small area A. Indeed, let us consider a Poincaré cycle for the return to the small area A (see Fig. 2). An open cycle can be turned into a closed one by an exponentially small stirring of the initial condition. The formula (2.5) demonstrates that there is an infinite number of ways for closing the trajectory. Therefore $N(T)$ is proportional to the envelope phase volume for the periodical orbits with a period $\leq T$. Hence the probability density of appearance of an orbit with a given period T is

$$P(T) \sim \text{const}/N(T), \tag{2.6}$$

and this should have the same magnitude as the probability density of the recurrence $f(T)$. Substituting (2.6) into (2.2) we obtain (2.3), (2.4). In fact, the distribution of the recurrences time has a more complex structure because the real systems with dynamical chaos are not equivalent to, for instance, a system of the Sinai's billiard type. The phase space of a dynamical system is strongly non-uniform. Within areas with chaotic dynamics there are islands which are not pen-

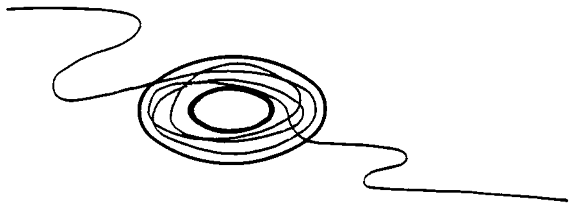

FIG. 3. A sketch of an orbit that gets stuck.

etrated by the trajectories from the chaotic area. The borders of the islands are sticky, i.e. having gotten into a very narrow band at the border, the trajectory gets stuck there for a long time (Fig. 3). As will be demonstrated below, this property of trajectories crucially influences the large time-scale asymptotics.

III. ERGODICITY AND STICKINESS

The idea that Poincaré used to prove his theorem of recurrences is rather simple and is based on the phase volume preservation for Hamiltonian systems. We would like to give a brief reminder of it, as this will help in understanding the possible consequences of stickiness. Consider a small area of the phase space A with volume Γ_A. For simplicity we can say that within A there is always an area A_1 with volume Γ_{A_1} consisting of particles that leave A within a limited time period t_1, since otherwise it would not have been necessary to prove the return of particles. Similar reasoning can be used in considering the exit of particles from the area A_1 which should occur within a certain time period t_2. Hence the particles assume a new position, i.e. the area A_2 with volume $\Gamma_{A_2}=\Gamma_{A_1}\neq 0$ and $A_1\cap A_2=0$, since otherwise the particles accumulate in A_1. If the particles performing a finite motion, never return to the initial area A, then $\Gamma_{A_1}+\Gamma_{A_2}+\cdots=\infty$ because all Γ_{A_i} are equal, which contradicts the condition of finiteness of motion. Hence the particle must return to A in a finite time T. The above reasoning demonstrates that a return time can not be less than an exit time from the area A which can be measured relatively easily. Let $\psi(\tau)d\tau$ be the probability that a particle exits the area A in a time τ which falls within the interval $(\tau,\tau+d\tau)$. Then the probability of the particle exiting the area A in the time $\leqslant t$ is

$$\Phi_e(t)=\int_0^t \psi(\tau)d\tau; \quad \Phi_e(\infty)=1 \tag{3.1}$$

(escape probability); and the survival probability is

$$\Phi_s(t)=\int_t^\infty \psi(\tau)d\tau=1-\int_0^t \psi(\tau)d\tau=1-\Phi_e(t);$$
$$\Phi_s(\infty)=0. \tag{3.2}$$

For the dynamical processes considered in Section II the magnitude $\psi(t)\sim f(t)$ and, consequently, decreases exponentially with time. However, the stickiness phenomenon can lead to a totally different pattern of returns or escapes. Numerous simulations[23,26–38] as well as the analysis of some simple models point at the existence of asymptotics,

$$\psi(t)\sim t^{-\beta-1} \quad (\beta>0, t\to\infty), \tag{3.3}$$

in real models of Hamiltonian chaos. In this case the formulas (3.1), (3.2) result in the following possibilities for the survival time t_s:

$$t_s\equiv\int_0^\infty \tau\psi_s(\tau)d\tau\approx\int_0^{\text{const}}\tau\psi_s(\tau)d\tau+\int_{\text{const}}^\infty \frac{\tau d\tau}{\tau^{\beta+1}}$$
$$=\begin{cases}\text{const}, & \beta>1,\\ \ln_{t\to\infty}t, & \beta=1,\\ \infty, & \beta<1.\end{cases} \tag{3.4}$$

In the case $\beta\leqslant 1$ the average survival time t_s is infinite. This result seems to have never been seriously discussed in terms of the Poincaré Recurrence Theorem. To avoid a seeming discrepancy one should understand the difference between individual events themselves and the frequencies of their occurrence (i.e. the probabilities). This is what Maxwell referred to in the quote given in the Introduction. Although a particle always returns to the initial area within a finite period of time (Poincaré Theorem), the frequencies of recurrence times can be such that an average recurrence time t_{rec} will be infinite ($t_{\text{rec}}>t_s=\infty$).

The existence of random processes with infinite moments, starting from a certain one, has been known since the St. Petersburg's paradox of Bernoulli was introduced (see, for example, Ref. 39) and has been studied for a vast class of phenomena known as Lévy's processes.[40] One can get a good idea of these processes by considering Weierstrass random walk introduced in Refs. 41–43 in which the size of a walking step can assume the values $x_n=a^n (n=1,2,\ldots)$ and the probability of the corresponding step is $p_n=\text{const}/b^n$ $(b>1)$. In that case

$$\langle x\rangle=\sum_{n=1}^\infty x_n p_n=\text{const}\sum_{n=1}^\infty\left(\frac{a}{b}\right)^n \tag{3.5}$$

and $\langle x\rangle=\infty$ if $a>b$. In a similar way one can easily devise processes for which divergence begins at higher moments.

If a particle moves at a constant speed, then $x=\text{const}\cdot t$ and we encounter a similarity of different time intervals of a process. This similarity as well as fractal properties of time sequences have been thoroughly discussed in a number of physical problems.[41,44,45] An analogous phenomenon seems to occur in the problems of dynamical chaos as well.[23–25] The peculiarities of dynamical chaos compared to a "conventional" stochastic process will be discussed below.

The fact that the distribution function (3.3) and the moments (3.5) have self-similarity properties results in an unusually slow convergence in the condition associated with the ergodic property of a system,

$$\overline{f(x(t))}\equiv\lim_{t\to\infty}\frac{1}{t}\int_0^t f(x(\tau))d\tau=\langle f(x(t))\rangle=\langle f(x(0))\rangle=f_0, \tag{3.6}$$

where f is an integrable function of the dynamical variables x; angular brackets mean that averaging is made over the phase volume of a system.

The usual way to observe the property (3.6) is to find the correspondent values $\bar{f}$ and $\langle f\rangle$ and to compare them. For example, phase averaging $\langle f\rangle$ can be performed by observing N orbits with the time span t_N each. One can expect that if

$$Nt_N=t \tag{3.7}$$

and t is sufficiently large, then the results for $\bar{f}$ and $\langle f\rangle$ in (3.6) will be approximately the same. The self-similarity of the random process of chaotic dynamics and the existence of traps in the phase space of a system give rise to new problems. The most typical question is[23] what the time scale t_0 is and what the sufficient number of orbits N_0 is such that for $t_N>t_0$ and $N>N_0$ the value of $\bar{f}$ is close to $\langle f\rangle$. If the pair (t_0,N_0) exists then one can use the condition (3.7) in different ways. Only for the systems with good mixing properties one can prove the fast convergence of $\bar{f}$ to $\langle f\rangle$ and the existence of the pair (t_0,N_0) using, for example, Markov's partition method.[46] As it was demonstrated in Ref. 23, slow convergence in (3.6) for dynamical chaos creates serious difficulties for the theory of chaos and a possibility of a new look at the MD problem.

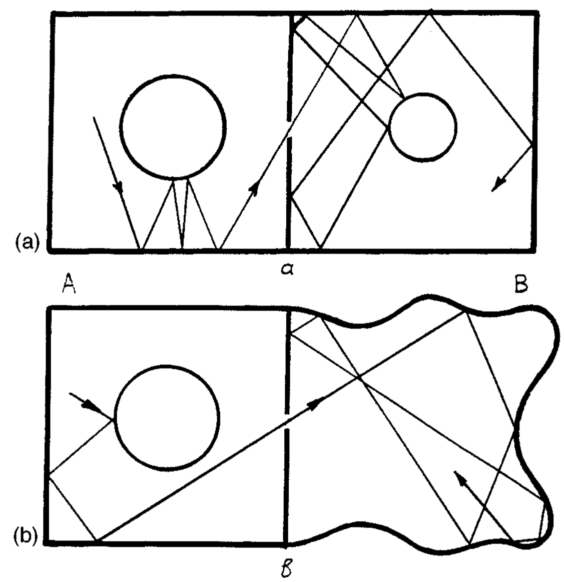

FIG. 4. Two billiards-type containers connected by a small hole in the division: (a) both billiards have good mixing properties (Sinai billiards); (b) one billiard (B) has a long power-like tail in time correlator for the trajectories, which creates time-traps.

IV. MAXWELL'S DEMON REVISED

In this section we would like to offer a new outlook of the MD problem based on the use of solely the dynamical chaos phenomenon. Working on the problem of statistical laws foundation, one should be striving to derive the main laws from the basic principles, and, similarly, considering the MD problem one would wish to devise not just the scheme of how MD is supposed to function, but the equations of motion (or the Hamiltonian) which would create automatically a non-equilibrium (from the thermodynamical viewpoint) state. At this point we will have to face the need to be able to track each individual trajectory, i.e. one has to be capable of doing the very thing which (according to Maxwell—see the quote in the Introduction) should have been rejected in the statistical approach to phenomena. It is important to understand that a thermodynamically non-equilibrium state can be created not only by achieving a temperature difference between the chambers A and B (Fig. 1), but also by achieving a pressure or density difference, or, in general, any difference between moments which are responsible for the macroscopic state of a system.

Let us explain how the dynamical chaos phenomenon leads to the equilibrium distribution of particles called invariant measure. Consider a modification of Fig. 1 in which chambers A and B are replaced with two different Sinai's billiards [Fig. 4(a)] communicating via a hole in the division and having equal volumes of the admissible areas of particle motion. The billiards A and B differ in that they have different mixing times, τ_A and τ_B, which can also be said about the relative average times, $\langle t_A\rangle/t$ and $\langle t_B\rangle/t$, of a single particle's stay in each of the chambers, when a total observation time $t\to\infty$.

According to the Poincaré Recurrence Theorem a particle can not get stuck in either of the chambers. As it was noted in Section II, the distribution of recurrence times has a Poissonian type and, therefore, the average recurrence time will be finite. The numerical analysis has shown that the following equality is true:

$$\frac{1}{t}\langle t_A\rangle=\frac{1}{t}\langle t_B\rangle \quad (t\to\infty), \tag{4.1}$$

independently of the type of relationship between the times τ_A, τ_B, if only there exist finite limits (4.1) at $t\to\infty$. This appears relatively easy to demonstrate. If one takes into account that collisions of a particle with the cushions of billiards A and B do not change the energy of the particle ($\mathbf{v}^2=v^2=$ const), then the condition (4.1) can be considered as an equality of temperatures $T_A=T_B$ for an arbitrary initial ensemble of particles that have different speeds and do not interact with each other.

The whole picture can be altered if, for instance, one arranges the billiard B in such a way that the islands, trappings, stickiness, etc. (described in Section III) will appear in the particle's phase space [see Fig. 4(b)]. Then it may turn out that

$$\lim_{t\to\infty}\frac{1}{t}\langle t_B\rangle=\infty \tag{4.2}$$

and the corresponding survival probability $\Psi_s(t)$ in (3.2) has a power tail at $t\to\infty$. In this case the equality (4.1) does not exist, since one of the limits is finite and the other one is infinite. We shall call this situation the dynamical model of MD (DMMD).

In fact, one can get a broader view of the problem by including the case

$$\lim_{t\to\infty}\frac{1}{t}\langle t_B^n\rangle=\infty, \quad n\geqslant 1. \tag{4.3}$$

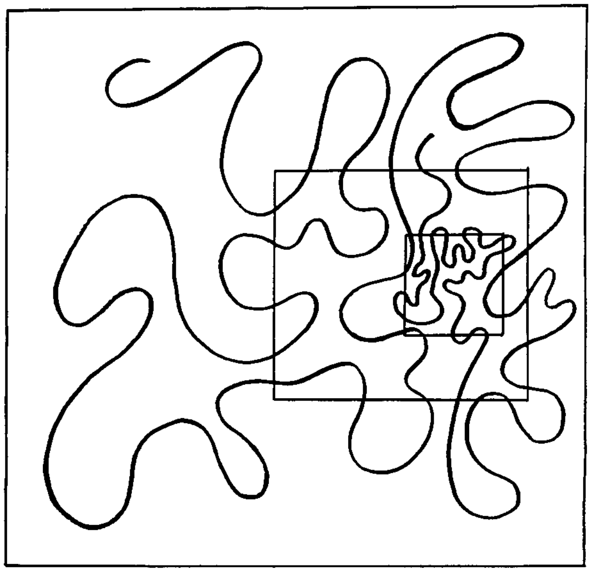

FIG. 5. A sketch of the recurrent time-trap (time-wrinkle in the phase space).

This case will correspond to the violation of certain thermodynamical conditions of equilibrium which do not have be the equalities of the temperatures or the densities.

V. STRETCHING-TIME DOMAINS

In the next section an example of DMMD is considered, while in this one we describe a scheme of the statistical trap corresponding to the stretching time. This scheme is known as one of the possible ways of fractal time realization for the Hamiltonian chaos[23–25] and is essentially analogous to the Weierstrass random walk in time.[41–45]

Consider the trajectory of a particle in the phase space. Let the trajectory corresponds, for example, to the finite area of chaotic dynamics. Select in the phase space an area corresponding to the γ-hypercube with side R_0. In Fig. 5, $\gamma \leqslant 2$. We assume that mixing and wondering in the phase space are almost uniform everywhere except for a certain area inside D_0. Then within D_0 one can isolate a subarea $D_1 \subset D_0$ with volume $D_1 = R_1^\gamma =$ such that a fine structure of wondering inside D_1 is uniform everywhere except for the area $D_2 \subset D_1$ with volume $D_2 = R_2^\gamma$ $(R_2 < R_1)$, etc.

The formal design of the stretching time is based on the fact that there exists a self-similarity of the areas D_n,

$$R_n = \lambda_R^n R_0 \quad (\lambda_R < 1) \tag{5.1}$$

and the time intervals T_n,

$$T_n = \lambda_T^n T_0 \quad (\lambda_T > 1), \tag{5.2}$$

spent by a particle in the area

$$D_n - D_{n+1} = D_n (1 - \lambda_R^\gamma) < D_n . \tag{5.3}$$

One should note that $\lambda_R > 1$ which corresponds to progressively longer periods of particle's being stuck, as the size of the area D_n decreases.

A particle randomly visits different areas D_n. Let us evaluate an average time $\langle T \rangle$ of particle's being stuck in the area D_0. Consider the equality

$$D_0 = (D_0 - D_1) + (D_1 - D_2) + \ldots + (D_n - D_{n+1}) + \ldots . \tag{5.4}$$

The probability that the particle gets into each of the areas $(D_n - D_{n+1})$ is proportional to its volume, i.e.

$$p_n = \text{const}(R_n^\gamma - R_{n+1}^\gamma) = \lambda_R^{\gamma n}(1 - \lambda_R^\gamma), \tag{5.5}$$

where the relationships (5.1), (5.3) and the normalization condition for p_n are used. Hence, according to (5.2), (5.4), (5.5) we obtain

$$\langle T \rangle = \sum_{n=0}^{\infty} p_n T_n = T_0 (1 - \lambda_R^\gamma) \sum_{n=0}^{\infty} \lambda_T^n \lambda_R^{\gamma n} . \tag{5.6}$$

At

$$\lambda_T \lambda_R^\gamma < 1, \tag{5.7}$$

we get

$$\langle T \rangle = T_0 \frac{1 - \lambda_R^\gamma}{1 - \lambda_T \lambda_R^\gamma}. \tag{5.8}$$

However, $\langle T \rangle$ diverges, if

$$\lambda_T \lambda_R^\gamma \geqslant 1, \tag{5.9}$$

with the case of equality being precisely correspondent to the Bernoulli paradox case.[42] Thus when (5.9) is true, the trapping of a particle in the area D_0 is possible in such a way that its mean average entrapment time in D_0 is infinite [compare with the condition (3.5)]. The condition (5.9) implies that

$$\mu = \gamma \ln(1/\lambda_R)/\ln \lambda_T \leqslant 1. \tag{5.10}$$

It remains now to be found if a system in which the dynamical trajectories behave in the above manner can exist, i.e. if there exist areas D_n with scaling relationships (5.1), (5.2) and the inequality (5.10) between scaling constants λ_R, λ_T is possible. An example of such a system is given in the next section.

VI. AN EXAMPLE OF THE DYNAMICAL MODEL OF MAXWELL'S DEMON (DMMD)

As an example of a dynamic system with chaotic dynamics in which the above phenomenon of the time stretching can occur, let us consider the problem of a perturbed pendulum,[47]

$$\ddot{x} + \omega_0^2 \sin x = -\epsilon \omega_0^2 \sin(x - \nu t), \tag{6.1}$$

where ν is frequency of the perturbation, ϵ is the dimensionless parameter of perturbation and ω_0 is the frequency of small oscillations of the unperturbed problem. The equation (6.1) has numerous applications (see the review in Ref. 48 and the most recent data[49]). Strictly speaking, this example does not define a closed system. Nevertheless the topological properties of the phase space of 1-1/2 degrees of freedom system (6.1) and 2 degrees of freedom system with time independent Hamiltonian are similar if some general conditions of nondegeneracy are applied. That is why the model

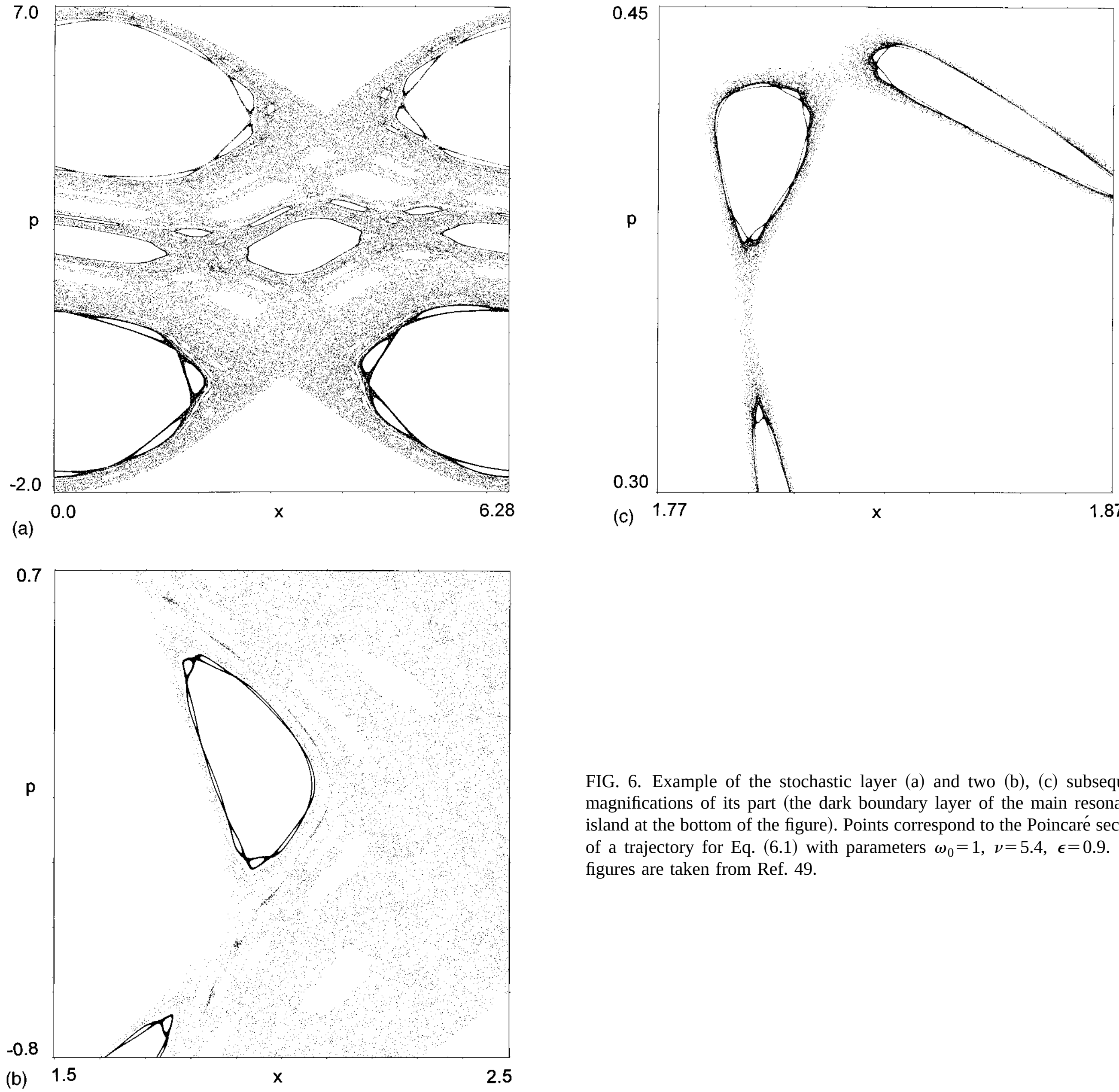

FIG. 6. Example of the stochastic layer (a) and two (b), (c) subsequent magnifications of its part (the dark boundary layer of the main resonance island at the bottom of the figure). Points correspond to the Poincaré section of a trajectory for Eq. (6.1) with parameters $\omega_0=1$, $\nu=5.4$, $\epsilon=0.9$. The figures are taken from Ref. 49.

(6.1), which is easier for a long time fine simulation, will be used to demonstrate the possibility of design of MD operation on the basis of a specific property of the phase space of a system with dynamical chaos.

It is known that a perturbation destroys the separatrix of the unperturbed pendulum and causes the occurrence of a stochastical layer with a very complex topological structure. The example of such layer is shown in Fig. 6(a) taken from Ref. 49. The points correspond to the Poincaré mapping of a single trajectory with mapping period $2\pi/\nu$. The darker areas are the aggregates of points and represent trappings of a particle near the island boundaries with the trajectory not penetrating into the islands. As it was noted above, stickiness of the islands boundaries is a general property of the Hamiltonian chaos, which is demonstrated by dark bands near the island boundaries in Fig. 6(a). In Fig. 6(a) the trajectory [or, more precisely, its Poincaré section for the cylinder phase space with x (mod 2π)] is demonstrated. As an example let us consider the lower island in greater detail. Its dark band consists of 7 subislands. One of the subislands is magnified in Fig. 6(b) in which subislands of the next generation and the area of the stickiness are seen. The magnification of the upper portion of Fig. 6(b) is shown in Fig. 6(c). It reveals persistence of the islands' generations and the stickiness of the island boundaries.

As a whole the picture of island formation and particle trapping near the island boundaries resembles the above system (see the previous section) of the areas D_n enclosed with one another. The results of Ref. 49 demonstrate the existence of the similarity laws (5.1), (5.2) with parameters

$$\lambda_R^2 \sim 0.07; \quad \lambda_T \sim 8.9. \tag{6.2}$$

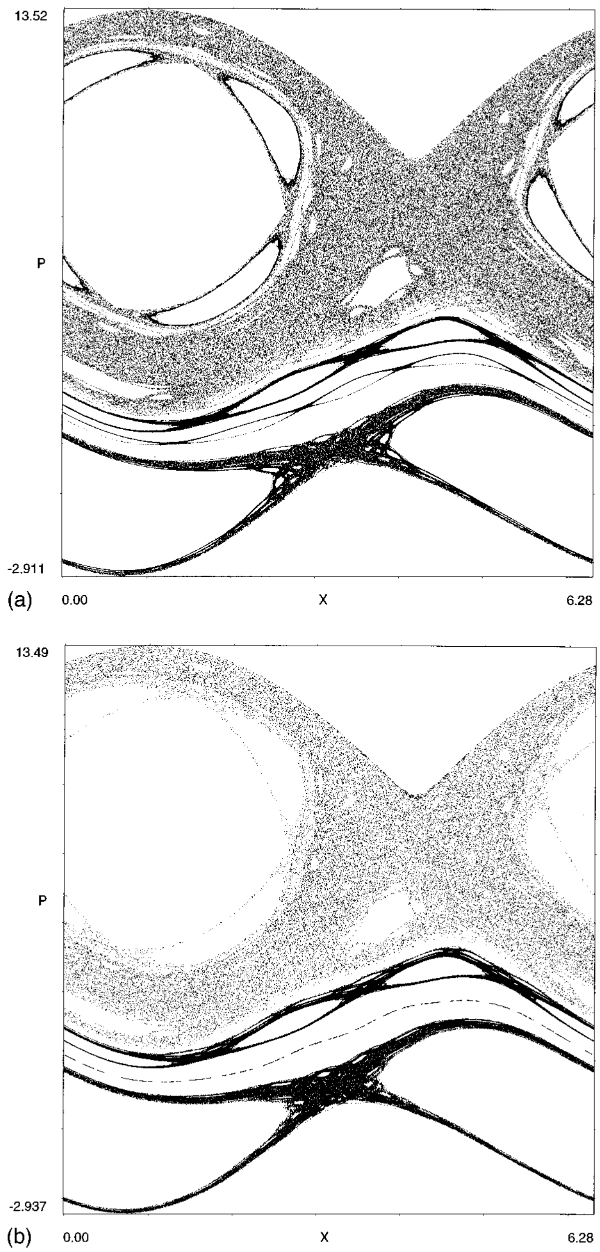

FIG. 7. Nonoverlapped (a) and two slightly overlapped (b) stochastic layers for the same model as in Fig. 6 with parameters $\omega_0=1$, $\epsilon=8.1$, and $\nu=8.6$ (a), 8.57 (b). The last one with $\nu=8.57$ can be used as a MD-operator. Overlapping is between the main (on the top stochastic layer and the next below it.

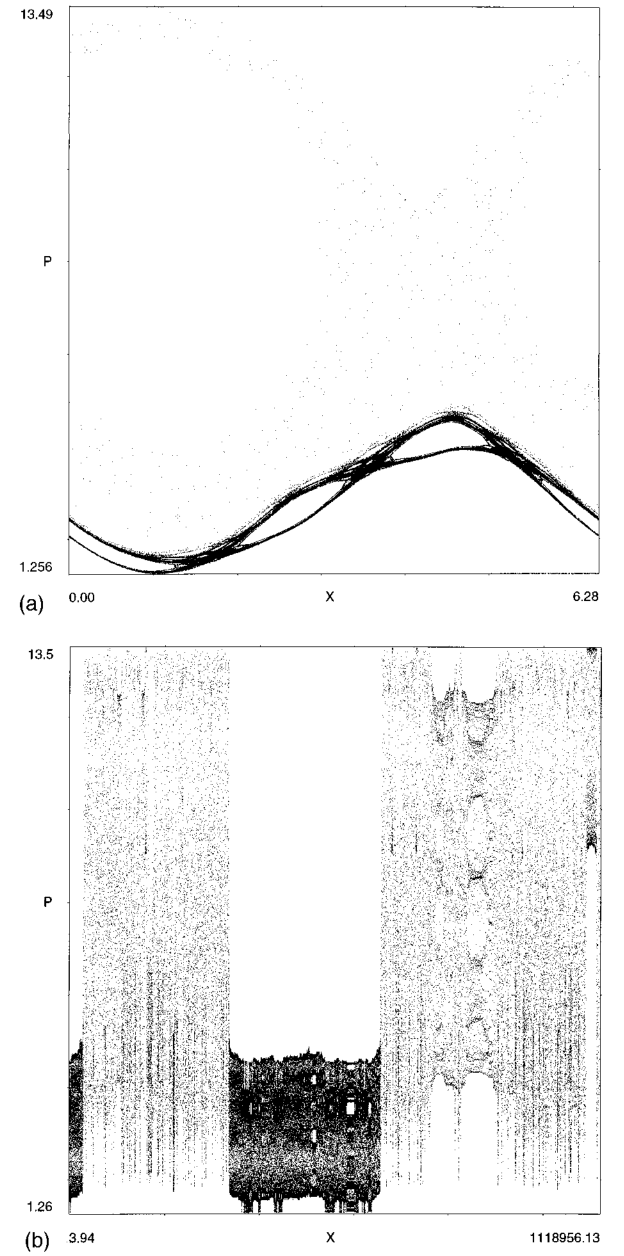

FIG. 8. Poincaré sections of an only orbit of the case in Fig. 7(b), which displays clearly the trajectory stickiness at the narrow layer of the picture's bottom. This part of the phase space can be considered as the chamber B whereas the top part of the phase space can be considered as the chamber A. There are two different displays of the orbit: (a) in the cylinder phase space $(x, \bmod 2\pi)$; (b) infinite in x in the phase space. The darkness indicates more visitings of the area.

Unfortunately, there is no reliable data for γ. We can assume that $\gamma\sim 1$, since particle's getting into the trapping area is associated simply with crossing the boundary of the stochastic layer. In reality γ can slightly exceed 1. Hence in accordance with the formulas (5.10), (6.2) one derives

$$\mu\sim \tfrac{1}{2}\ln(1/0.07)/\ln 8.9\sim 0.6<1,$$

and the condition is met that the boundary area (annulas) around the principal island in Fig. 6 can be considered as a trap with infinite average lifetime.

The practical realization of MD based on the above dynamical model is the following. For some values of parameters ϵ and ν/ω_0 in (6.1) there exists a series of statistical layers separated by invariant curves [Fig. 7(a)]. By slightly changing the value of ϵ one can cause, say, the two upper layers to merge or even slightly overlap [Fig. 7(b)]. Then the lower stochastic layer can be considered as a chamber B, while the upper one—as a chamber A. Figure 8(b) representing Poincaré section for one trajectory, demonstrates a long term trapping in the chamber B and supports the relevance of introducing the two chambers A and B with different particle density. As the case illustrated by Fig. 8(b) is related to the

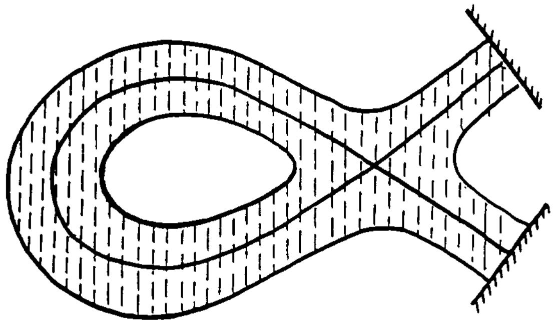

FIG. 9. A sketch of the so-called ergodic divertor with two divertor plates for the tokamak plasmas.

phase plane, the average energies of particles in the areas A and B also differ.

The actual physical meaning of the described situation is that chaotic dynamics leads neither to the uniform mixing in phase space nor to the uniform relaxation and transport. The mixing process describes a small scale behavior, while the transport process (kinetics) describes a large scale one. This statement refers to the typical cases of Hamiltonian chaos. There is no rigorous theory of the anomalously sticky domains in the phase space, so the asymptotic properties of the phenomenon of island boundary stickiness are absent. Nonetheless one can argue that for some ranges of the perturbation parameter value the average trapping time can turn out to be comparable to any large value of the time interval of the system's existence or observation. Thus, for this interval and in the mentioned above limited sense the chaotic dynamics can give rise to the thermodynamically non-equilibrium state.

VII. EXAMPLES

Two examples of the phenomena related to DMMD will be given below. It was shown that the dynamical design of MD is based on a very specific interplay between the space and time self-similarity of a dynamical process. The process consists in that orbits go around a set of self-similar islands and switch from one island set to another. The rotations around small islands resemble the rotations around bigger islands so that one can speak about the persistence of the space memory. For the same process the smaller the island the longer the period of revolution around it. This property of the dynamical process was named time-stretching and its simultaneous connection to the space self-similarity reminds us of the famous painting by Salvator Dali entitled ''Persistence of memory'' which depicts a soft (stretching) clock. (Dali's own commentary to this painting is: ''Materialisation of the flexibility of time and the indivisibility of time and space. Time is not rigid. It's one with space—fluid.''[51])

The two examples below are related to the DMMD.

(a) *Plasma edge of the tokamak*. It is supposed that in the fusion reactor of the tokamak type the energetic particle should load a huge energy to the thin layer of plasma edge located near a separatrix of the magnetic field of the device. To make the energy loading more ''smooth'' an ergodic divertor (Fig. 9) is used which creates a stochastic layer near the plasma edge. As it was shown above by a special choice of the field of perturbation one can find the conditions which will make it possible to separate and trap necessary (or unnecessary) groups of ions for a fairly long time. In this model the bulk plasma can be considered as a chamber A and the part of the edge plasma plays the role of a chamber B (high temperature area).

(b) *Finite-time computability*. Let us consider a computational algorithm and raise the question of its finite time computability in the described below sense. The higher the accuracy of the algorithm the longer the computation time. In performing a computation with an adaptive grid one has to deal with a probabilistic measure of the grid sizes that occur during prolonged computations. What is the mean time τ necessary to perform the computation with an appropriate accuracy?

The problem in a very rough sense possesses a feature of the Weierstrass-like random walk described in Section III. Indeed let the sizes of grids follow the exponential law

$$r_n \sim \lambda_r^n r_0 \quad (\lambda_r < 1), \tag{7.1}$$

and let the computational time for a grid with size r_n is t_n with the same kind of the exponential law,

$$t_n \sim \lambda_t^n t_0 \quad (\lambda_t > 1). \tag{7.2}$$

From (7.2) and (7.1) we have, by excluding n,

$$t_n / t_0 \sim (r_0 / r_n)^{\mu}, \tag{7.3}$$

with

$$\mu = \frac{\ln \lambda_t}{\ln 1/\lambda_r} > 0. \tag{7.4}$$

It is obvious that $\xi_n = r_0 / r_n > 1$ and the behavior of $\tau_n = t_n / t_0$ depends on the value of μ and, generally speaking, on the distribution function $f(\xi)$ of how often different grids occur during the computations. If $f(\xi) \sim \exp(-\xi/a)$ or $\exp(-\xi^2/a^2)$ then there exists a finite scale a which cuts off the integral for any moment $\langle \xi^m \rangle$ and, consequently, $\langle \tau \rangle = \langle t_n / t_0 \rangle$ is finite. If, for example,

$$f(\xi) \sim 1/\xi^{1+\alpha} \quad (\xi \to \infty, \text{or } \ r_n \to 0), \tag{7.5}$$

then

$$\langle \tau \rangle \sim \langle \xi^{\mu} \rangle = \begin{cases} \text{const} & (\mu < \alpha), \\ \infty & (\mu \geqslant \alpha). \end{cases} \tag{7.6}$$

The expression (7.6) gives a practical rule for an adaptive grid scheme of computations which consists in that for a fixed α the integrator should satisfy the condition

$$\ln \lambda_t / (\ln 1/\lambda_r) < \alpha, \tag{7.7}$$

which means simply that ''there is no presence'' of the MD in our computations.

VIII. CONCLUSIONS

In this article we have discussed two issues related to the principal problem of the Hamiltonian dynamical chaos which is to determine what the real chaotic trajectory is. The first issue is related to the role of islands in the phase space of a system, and the second one reformulates the near-island dynamics process as a possible model for Maxwell's Demon (MD). Crucial for the chaotic dynamics is that the large time asymptotics for the majority of typical orbits is still outside our understanding while the short time scale properties such as the sensitivity to the initial condition perturbation are reasonably clear. We can modify the formulation of difficulties by simply mentioning that some fine properties of the small time scale dynamics, being behind the stage, strongly influence the large time asymptotics. Our conjecture is that the large time asymptotics of the kinetics is determined by the motion entangled in islands-around-islands domain . In a recent publication[51] it was found by a sophisticated simulation of the standard map that for at least 10^{10} steps the dynamics corresponds to the anomalous (non-Gaussian) statistics.

We believe that some irregularities of the chaotic dynamics, being different from those of the normal (Gaussian) dynamics, could be critical for the general ambitious goal of the statistical laws foundation. In this article we have shown, that if the fractional kinetics exists for the Hamiltonian chaos situation, then it could manifest itself in the possibility to design MD. Even if the fractal time process is only a suitable approximation for fairly long time periods, it is sufficient to introduce a ''restricted'' MD which will operate for a time period comparable to the lifetime of the universe.

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