Changes in the adiabatic invariant and streamline chaos in confined incompressible Stokes flow

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The steady incompressible flow in a unit sphere introduced by Bajer and Moffatt [J. Fluid Mech. **212**, 337 (1990)] is discussed. The velocity field of this flow differs by a small perturbation from an integrable field whose streamlines are almost all closed. The unperturbed flow has two stationary saddle points (poles of the sphere) and a two-dimensional separatrix passing through them. The entire interior of the unit sphere becomes the domain of streamline chaos for an arbitrarily small perturbation. This phenomenon is explained by the nonconservation of a certain adiabatic invariant that undergoes a jump when a streamline crosses a small neighborhood of the separatrix of the unperturbed flow. An asymptotic formula is obtained for the jump in the adiabatic invariant. The accumulation of such jumps in the course of repeated crossings of the separatrix results in the complete breaking of adiabatic invariance and streamline chaos. © *1996 American Institute of Physics.* [S1054-1500(96)01001-2]

I. INTRODUCTION

Streamline chaos in steady incompressible flow has attracted considerable attention in connection with impurity transport (see, for example, Refs. 1 and 2) and the dynamo problem.[3] Steady flow with a quadratic velocity field and chaotic streamlines within a bounded invariant domain (a sphere) was discussed in this context in Refs. 4–6. The properties of this flow were examined in the investigation reported in the present paper.

We shall consider the steady incompressible flow whose streamlines are described by the following set of differential equations:

$$\begin{aligned} \dot{x} &= -8xy + \alpha z, \\ \dot{y} &= 11x^2 + 3y^2 + z^2 - 3, \\ \dot{z} &= 2zy - \alpha x. \end{aligned} \tag{1.1}$$

The unit sphere $x^2 + y^2 + z^2 \leqslant 1$ remains invariant for the flow defined by (1.1). We note that the flow satisfies the Stokes equations.[6] We shall be interested in small values of the parameter α ($0 < \alpha \ll 1$). The streamlines (1.1) are regular for $\alpha = 0$ (almost all the streamlines lying inside the unit sphere are closed[6]). It was shown numerically in Ref. 6 that the flow (1.1) has a remarkable property: for arbitrarily small $\alpha \neq 0$ the entire unit sphere is the domain of streamline chaos (the unbounded continuation of a typical streamline will densely fill the entire unit sphere). This property distinguishes the above flow from many other flows produced by a small perturbation of a flow with regular streamlines. In previously considered examples of such flows, the size of the region of streamline chaos tends to zero as the size of the perturbation tends to zero.[7]

The averaging method[8] can be used in an approximate description of streamlines in the system defined by (1.1). The averaged system has an integral which, for time intervals of the order of $1/\alpha$, is an approximate integral for the exact streamlines (1.1), i.e., it is an *adiabatic invariant* (AI) [note that the *time* in this statement is the independent variable in (1.1)]. Numerical integration shows[6] that the AI undergoes a jump along a streamline when the latter crosses a small neighborhood of the separatrix of the unperturbed ($\alpha = 0$) system. The accumulation of such jumps in the course of multiple crossings of the separatrix gives rise to the diffusion of the AI and to the onset of streamline chaos.

We shall derive an asymptotic formula for the jump in the AI due to the separatrix crossing in the system (1.1). The jump is proportional to $\alpha^{3/4}$ and is quasirandom: it is very sensitive to small changes in the initial conditions. Jumps of this kind give rise to the diffusion of the AI by an amount of order 1 in a time of order $\alpha^{-5/2}$. This time is actually the time constant for the diffusion of a passive impurity (or scalar field) in the above system.

There is a well-known formula for the jump in the AI due to the separatrix crossing in a Hamiltonian system with one degree of freedom that depends on a parameter that is a slowly varying function of time.[9–11] For the systems discussed in Refs. 9–11, the separatrix of the unperturbed problem consists of trajectories passing through a family of degenerate singular points. For (1.1), the separatrix of the unperturbed problem consists of trajectories passing through nondegenerate singular points.

In Sec. II we briefly consider the properties of the unperturbed system. The averaging method is used in Sec. III to describe perturbed flow. The adiabatic invariant of the system is also introduced in Sec. III. An asymptotic expansion is obtained in Sec. IV for the adiabatic invariant in the neighborhood of the separatrix, and a formula is derived for the jump in the AI due to the separatrix crossing. The question of diffusion of the adiabatic invariant in the course of multiple crossings of the separatrix is discussed in Sec. V. Some derivations of formulas, and estimates of accuracies, are given in Appendices A–C.

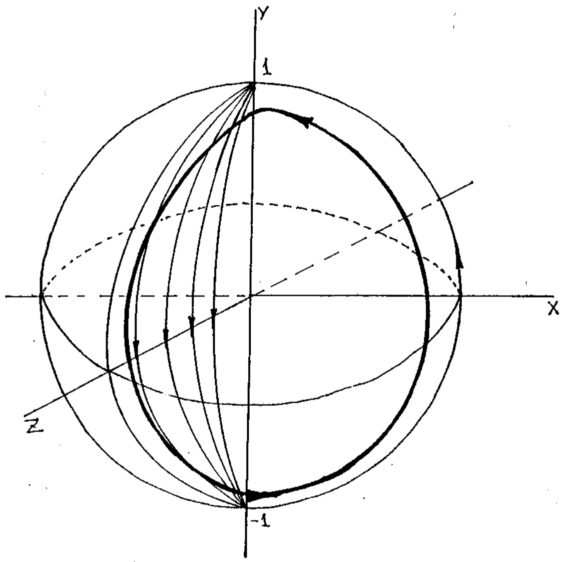


FIG. 1. Schematic representation of the streamlines of (1.1) in the absence of perturbation.

II. THE UNPERTURBED SYSTEM

The unperturbed ($\alpha=0$) system (1.1) has two integrals of motion[6]

$$I=xz^4, \quad J=\frac{x^2+y^2+z^2-1}{z^3}. \tag{2.1}$$

The $I=$const surface is a cylinder whose generator is parallel to the y axis. The $I=0$ surface consists of the $x=0$ and $z=0$ planes. The $J=$const surfaces are surfaces of revolution around the z axis. The $J=0$ surface is a unit sphere centered on the origin of coordinates.

The intersection of the $I=$const and $J=$const surfaces is an integral curve (streamline) of the unperturbed system. All the streamlines within the unit sphere, other than those passing through the poles of the sphere, are closed (Fig. 1). The unperturbed system can be described in the following form in terms of the integrals I and J:

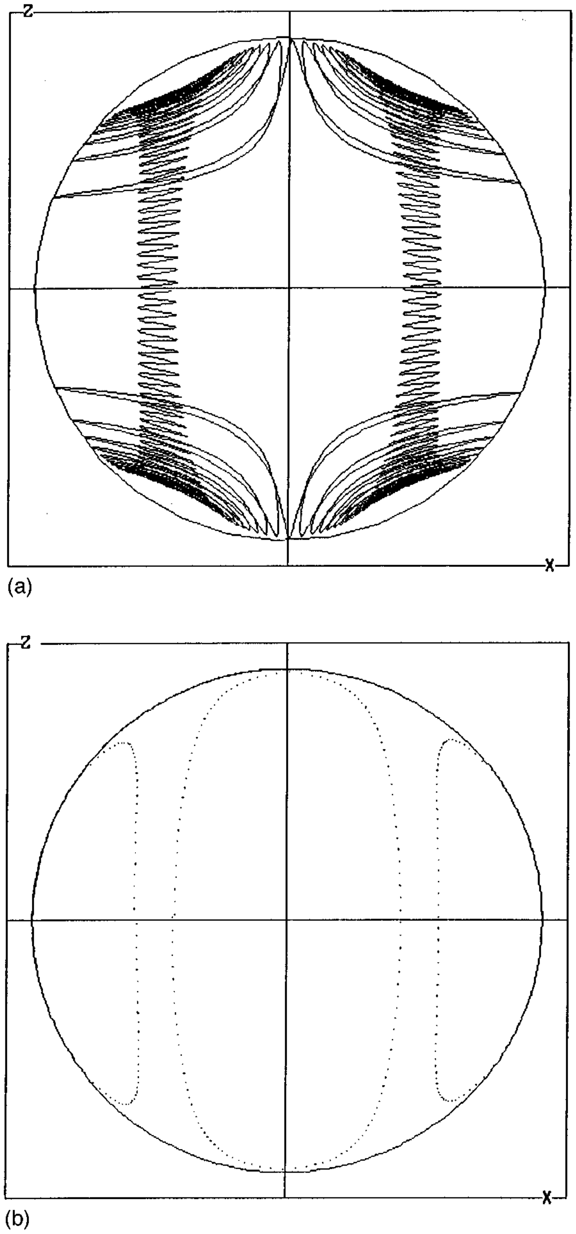
$$\dot{\mathbf{x}}=[\text{grad } I, \text{grad } J], \tag{2.2}$$

where $\mathbf{x}=(x,y,z)$ and the brackets indicate a vector product. Systems such as (2.2) are referred to as Nambu systems.[12]

The integral J is undefined on the $z=0$ plane. In the neighborhood of this plane, we can use another set of integrals for $x\neq 0$, namely,

$$L=|I|^{1/4} \text{ sign}(x), \quad K=|I|^{3/4}J \text{ sign}(x). \tag{2.3}$$

The poles of the unit sphere, $x=z=0$, $y=\pm 1$, are singular saddle points of the unperturbed system with eigenvalues $\lambda_1=\mp 8$, $\lambda_2=\pm 6$, $\lambda_3=\pm 2$. The north ($y=+1$) and south ($y=-1$) poles are joined by invariant curves that fill part of the invariant plane $x=0$ that lies in the interior of the invariant sphere. This part of the $x=0$ plane will be called the separatrix. Two invariant curves run from the south pole to the north pole, namely, the semicircles $z=0$, $x^2+y^2=1$, sign $(x)=\pm 1$ (Fig. 1). The system also has degenerate singular points that fill the ellipse $y=0$, $11x^2+z^2=3$.



(a)

(b)

FIG. 2. Trajectories of the perturbed system (1.1) for $\alpha=0.1$: (a) projection onto the x,z plane, (b) section by the $y=0$ plane.

III. DESCRIPTION OF PERTURBED FLOW BY THE AVERAGING METHOD. ADIABATIC INVARIANT

For nonzero values of α, the streamlines (1.1) are not closed and cross the separatrix. Figure 2 shows a segment of a streamline (1.1) that crosses the separatrix twice (for $z<0$ and for $z>0$). Figure 2(a) shows the projection of the segment onto the (x,z) plane, whereas Fig. 2(b) shows the section of it by the $y=0$ plane.

Approximate descriptions of the flow (1.1) for small $\alpha>0$ can be based on the averaging method.[8] A point inside the unit sphere that does not lie on the separatrix or the $z=0$ plane can be specified by the coordinates i,j,φ where i,j are the values of the integrals (2.1) and φ (mod 2π) is the phase (angle variable) along the unperturbed streamline passing through the point under consideration. The perturbed system (1.1) has the following form in terms of the new variables:

$$\dot{i}=\alpha f(i,j,\varphi),$$

$$\dot{j}=\alpha g(i,j,\varphi),$$

$$\dot{\varphi}=\omega(i,j)+\alpha h(i,j,\varphi).$$

The functions $f,\ g,\ h$ have a period of 2π in φ. We now define the averaged system:

$$\dot{i}=\alpha F(i,j),\quad \dot{j}=\alpha G(i,j), \tag{3.1}$$

where F and G are the averages of f and g over φ. It follows from this definition that

$$\begin{aligned} F(i,j)&=\frac{1}{T(i,j)}\oint(\text{grad }I,\mathbf{v})dt,\\ G(i,j)&=\frac{1}{T(i,j)}\oint(\text{grad }J,\mathbf{v})dt, \end{aligned}\tag{3.2}$$

where $\mathbf{v}=(z,0,-x)$ is the perturbation vector field in (1.1), the integrals in (3.2) are evaluated along a trajectory (streamline) of the unperturbed system with $I=i,\ J=j$, and $T(i,j)$ is the period of the unperturbed motion along this trajectory. The solutions of the averaged system describe the behavior of the functions I and J along the trajectory of the original system (1.1) far from the separatrix to an accuracy of order α over time intervals of order $1/\alpha$.[8,13]

Let $\Phi(i,j)$ represent the flux of the perturbation vector $\mathbf{v}$ across a surface spanning the trajectory of the unperturbed system $I=i,\ J=j$. This flux depends on the values of the integrals i,j on this trajectory and is independent of the choice of the surface spanning it, since div $\mathbf{v}=0$. For a correct definition of the flux we must also choose the positive direction of the normal to the surface (this choice is discussed at the end of this section). Thus

$$\Phi(i,j)=\int_S(\mathbf{v},\mathbf{n})d\sigma, \tag{3.3}$$

where S is a surface spanning the unperturbed trajectory and $\mathbf{n}$ and $d\sigma$ are, respectively, the unit normal on S and an area element on S. The following formulas are derived in Appendix A:

$$\begin{aligned} F(i,j)&=\frac{1}{T(i,j)}\frac{\partial\Phi(i,j)}{\partial j},\\ G(i,j)&=-\frac{1}{T(i,j)}\frac{\partial\Phi(i,j)}{\partial i}. \end{aligned}\tag{3.4}$$

The averaged system is thus a Hamiltonian system with form of volume $T(i,j)dj\wedge di$ and Hamiltonian $\Phi(i,j)$. In particular, the function $\Phi(i,j)$ is an integral of the averaged system.

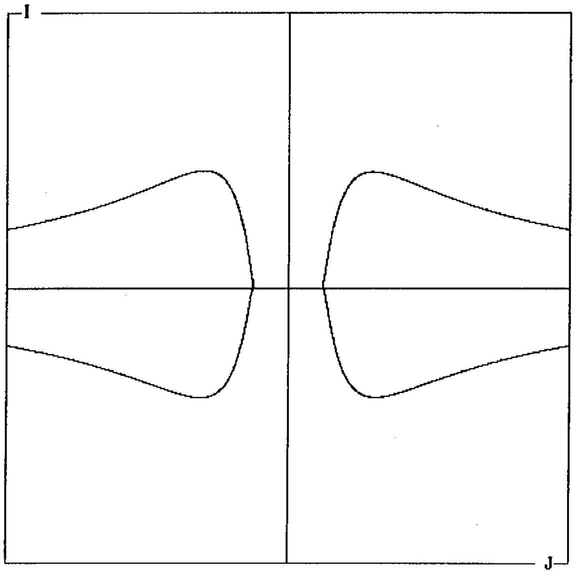


FIG. 3. Trajectory of the averaged system (3.1) projected onto the I,J plane for $\alpha=0.01$.

The basic fact for the ensuing discussion is that Φ is an integral of the averaged system; this can be deduced from the incompressibility of the flow (1.1) in a very straightforward way without using (3.4) (see Appendix A).

The phase portrait of the averaged system on the (i,j) plane, obtained numerically, is shown in Fig. 3. The departure of the trajectories to infinity in j is a fictitious singularity associated with the particular choice of the variables in the averaged system. The singularity can be removed by transforming to new variables $L,\ K$ defined by (2.3). In terms of $i,\ j$, on the other hand, the phase point takes a finite time to pass $j=\infty$ on its way from the right half-plane to the left half-plane and back again at constant Φ.

Trajectories of the unperturbed system lying on the separatrix correspond to points on the $i=0$ line in Fig. 3. The averaged system is not defined on the $i=0$ line, but it can be extended by continuity. The phase points of the averaged system reach the $i=0$ line in a finite time, and then leave it.

Along a streamline of the perturbed system (1.1), but far from the separatrix, the function Φ undergoes only oscillations of order α over times of order $1/\alpha$ (since Φ is an integral of the averaged system). It follows that Φ is an adiabatic invariant.[14] It is clear from Fig. 3 that all the $\Phi=$const lines cross the $i=0$ line that corresponds to the separatrix. The behavior of Φ along a streamline of the perturbed system in the neighborhood of the separatrix is discussed in Sec. IV.

We must now consider the choice of the positive direction of the normal to the surface whose edge is a trajectory of the unperturbed system. On this surface, the natural direction of rotation is specified by the unperturbed motion over the edge. If $I>0$ on the edge, the positive direction of the normal is taken to be the direction of the angular velocity, whereas for $I<0$ the opposite direction is taken. This choice of the direction of the normal enables us to extend the definition of $\Phi(i,j)$ to $i=0$ by continuity.

To evaluate the adiabatic invariant, we use (3.3) and the formula

$$\Phi(i,j)=\oint_{\partial S}(\mathbf{A},d\mathbf{l}), \tag{3.5}$$

where $\alpha\mathbf{A}$ is the perturbation vector potential, curl $\mathbf{A}=\mathbf{v}$, $d\mathbf{l}=(dx,dy,dz)$, ∂S is the edge of the surface S, i.e., the trajectory of the unperturbed system, and the direction of integration in (3.5) is the direction of the unperturbed motion along ∂S. We shall take $\mathbf{A}=\frac{1}{2}(0,x^2+z^2,0)$. Then, using the expression for the integral J given by (2.1), we obtain

$$\Phi(i,j)=-\frac{1}{2}\,j\oint_{\partial S}z^3\,dy. \tag{3.6}$$

This expression is used in Appendix B to determine the asymptotic behavior of $\Phi(i,j)$ in the neighborhood of the separatrix.

IV. CHANGE IN THE ADIABATIC INVARIANT DUE TO THE SEPARATRIX CROSSING

In this section, we derive the basic formula for the change in the adiabatic invariant due to the separatrix crossing.

A. Asymptotic expansion for the adiabatic invariant near the separatrix

For small $i>0$, the contour $\Gamma_{i,j}=\{I=i,J=j\}$ is close to the contour (with breaks) $\Gamma_{0,j}^{+}$ which consists of two smooth curves, namely, Γ_j which runs from the north pole to the south pole in the $x=0$ plane, on which $J=j$, and the semicircle $S^{+}=\{z=0,x^2+y^2=1,x>0\}$ running from the south pole to the north pole. Similarly, for small $i<0$, the contour $\Gamma_{i,j}$ is close to the contour with breaks $\Gamma_{0,j}^{-}$ consisting of the curve Γ_j and the semicircle $S^{-}=\{z=0,x^2+y^2=1,x<0\}$ running from the south to the north pole.

The adiabatic invariant $\Phi(i,j)$ has a singularity at $i=0$. It is shown in Appendix B that

$$\Phi(i,j)=\Phi_0(j)+\tfrac{4}{3}a(j)|i|^{3/4}+O(|i|^{5/4}), \tag{4.1}$$

where $\Phi_0(j)$ and $a(j)$ are smooth functions of j and $\Phi_0(j)$ is the flux of the vector $\mathbf{v}$ across a surface spanning the bounding contour $\Gamma_0^{\pm}$ (the fluxes of $\mathbf{v}$ through surfaces spanning $\Gamma_{0,j}^{+}$ and $\Gamma_{0,j}^{-}$ are equal).

Let $\Theta=\Theta(j)=\partial\Phi_0/\partial j$. We then find from (3.2) and (3.4) that

$$\Theta=\oint_{\Gamma_{0,j}^{\pm}}(\text{grad } I,\mathbf{v})dt. \tag{4.2}$$

This integral is evaluated along the contour $\Gamma_{0,j}^{\pm}=\Gamma_j\cup S^{\pm}$. However, the integrand is equal to zero on the curves $S^{\pm}$. It follows that we can replace $\Gamma_{0,j}^{\pm}$ with Γ_j in (4.2). The integral (4.2) is improper (an infinite time is required to reach a stationary point), but converges because grad I vanishes at stationary points.

It is shown in Appendix B that

$$\Theta=\oint_0^{z_m}\frac{z^4\,dz}{\sqrt{jz^3-z^2+1}}, \tag{4.3a}$$

$$a=-j\,\frac{3\sqrt{\pi}}{8}\,\frac{\Gamma(5/8)}{\Gamma(9/8)}, \tag{4.3b}$$

where z_m is the maximum absolute value of z along Γ_j. This maximum value is obtained by setting to zero the denominator of the integrand in (4.3a). It follows from (2.1) and (4.3) that the sign of Θ is opposite to that of j. Hence the solutions of the averaged system pass from the domain $i<0$ to the domain $i>0$ for $j<0$ and then back again for $j>0$ (cf. Fig. 3).

B. Preliminary description of perturbed motion

Consider a segment of a streamline of the perturbed system that crosses (once) the separatrix. Let M_- and M_+ be its initial and final points lying at a distance of order 1 from the separatrix. We shall use $i_\pm, j_\pm, \Phi_\pm$ to denote the values of $I,\ J,\ \Phi$ at the points $M_\pm$. To be specific, we shall suppose that the point M_- lies in the hemisphere with $x<0$ (and hence $i_-<0$) and M_+ lies in the hemisphere $x>0$ (correspondingly, $i_+>0$). The particular streamline segment crosses the separatrix for $j<0$. The problem is to calculate the asymptotic behavior of $\Delta\Phi=\Phi_+-\Phi_-$ in α.

Consider a fixed point M_- and an interval $[\tau_-,\tau_+]$ of slow time $\tau=\alpha t$. The streamline is a spiral, each turn of which is close to one of the (closed) streamlines of the unperturbed flow. We now fix our attention on successive points along a streamline at which $\dot{y}=0$, $y>0$ (the uppermost points). We shall use M_k to represent these points and t_k to represent the times at which they are crossed. The points will be numbered so that $k\leqslant 0$ for points in the $x<0$ hemisphere and $k>0$ for points in the $x>0$ hemisphere; increasing $|k|$ will correspond to a departure from the separatrix. Thus, M_0 is the last point of maximum rise prior to the crossing of the separatrix and M_1 is the first point of maximum rise after crossing of the separatrix. The turn of the streamline between the points M_{k-1} and M_k will be called its kth turn and will be denoted by γ_k. We shall use $\tilde{M}_k$ to represent the point at which $\dot{y}=0$, $y<0$ on this turn, and $\tilde{t}_k$ will represent the time at which the point $\tilde{M}_k$ is crossed.

We shall also use $i_k,\ j_k,\ \Phi_k$ to represent the values of $I,\ J,\ \Phi$ at the point M_k, $\Theta_0=\Theta(j_0)$, $a_0=a(j_0)$. In the case we are considering, $j_0<0$. Consequently, according to (4.3), $\Theta_0>0$. It can be shown that

(1) the quantity i_0 satisfies the inequalities

$$0>i_0>-\alpha\Theta_0-c\alpha^{7/4};$$

(2) if

$$-c\alpha^{7/4}>i_0>-\alpha\Theta_0+c\alpha^{7/4}, \tag{4.4}$$

then

$$i_1=i_0+\alpha\Theta_0+O(\alpha^{7/4})>\frac{c}{2}\,\alpha^{7/4},\quad j_1=j_0+\alpha O(i_1^{-1/4}),$$

where c is a positive quantity.

The proof is based on the fact that i increases on a turn near the separatrix by the amount $\alpha\Theta_0+O(\alpha^{7/4})$ provided the phase point does not pass too closely to stationary points (poles of the sphere) at which it may be delayed for a long time. The condition given by (4.4) ensures that the trajectory does not approach the poles too closely. We shall assume in the ensuing analysis that this condition is satisfied.

C. Formula for the change in the adiabatic invariant

Differentiating $I,\ J$ and using (1.1), we obtain

$$\dot{I}=\alpha(\text{grad } I,\mathbf{v})=\alpha z^3(z^2+3x^2), \tag{4.5}$$

$$\dot{J}=\alpha(\text{grad } J,\mathbf{v})=3\alpha x\,\frac{(x^2+y^2+z^2-1)}{z^4}=3\alpha J\,\frac{x}{z}. \tag{4.6}$$

Consider the turn γ_k with small i_k (i.e., near the separatrix). It follows from (4.5) that the change in I along γ_k occurs mainly on that part of the turn that lies near the $x=0$ plane (for $t_{k-1}\leqslant t\leqslant\tilde{t}_k$); the other part of the turn lies near the $z=0$ plane where $\dot{I}$ is small. Integrating (4.5) along γ_k and approximating integration along γ_k with integration along $\Gamma^{\pm}_{0,j}$, we obtain (see also Appendix C):

$$i_k-i_{k-1}=\alpha\Theta_0+\alpha O(|i_k|^{3/4})+O(\alpha^2). \tag{4.7}$$

It follows from (4.6) that the change in J along γ_k occurs mainly near the $z=0$ plane (for $\tilde{t}_k\leqslant t\leqslant t_k$), since $\dot{J}$ vanishes at $x=0$ and has a singularity at $z=0$. Integrating (4.6) along γ_k, and approximating integration along γ_k with integration along Γ_{j0}, and also using (3.2), (3.4), and (4.1), we find that

$$j_k-j_{k-1}=-\alpha a_0|i_k|^{-1/4}\,\text{sign}(i_k)+\alpha O(|i_k|^{1/4})+\alpha^2 O(|i_k|^{-1/2}). \tag{4.8}$$

For values of the function Φ on γ_k we find from (4.1) that

$$\Phi(i,j)=\Phi_0(j_0)+\Theta_0(j-j_0)+\tfrac{4}{3}a_0|i|^{3/4}+O(|i|^{5/4}+|j_0-j|^2+|j_0-j||i|^{3/4}). \tag{4.9}$$

From (4.7)–(4.9) it follows that

$$\begin{aligned}\Phi_k-\Phi_{k-1}={}&a_0(-\alpha\Theta_0|i_k|^{-1/4}\,\text{sign}(i_k)+\tfrac{4}{3}|i_k|^{3/4}\\&-\tfrac{4}{3}|i_{k-1}|^{3/4})+O(|i_k|^{5/4}+|i_{k-1}|^{5/4}\\&+|j_0-j_k|^2+|j_0-j_{k-1}|^2+|j_0-j_k||i_k|^{3/4}\\&+|j_0-j_{k-1}||i_{k-1}|^{3/4}).\end{aligned} \tag{4.10}$$

The change in Φ over a turn lying well away from the separatrix (at a distance ~ 1 from it) is $O(\alpha^2)$ since Φ is an adiabatic invariant. The total change in Φ over such turns is $O(\alpha)$. Consequently, the lower-order term in the asymptotic expression for the total change in Φ (if it exists) is due to turns lying near the separatrix. Hence $\Delta\Phi$ can be calculated in the leading approximation by using (4.7), (4.8), and (4.10), retaining only the leading terms in these expressions. The final result is as follows (the remainder term is estimated in Appendix C):

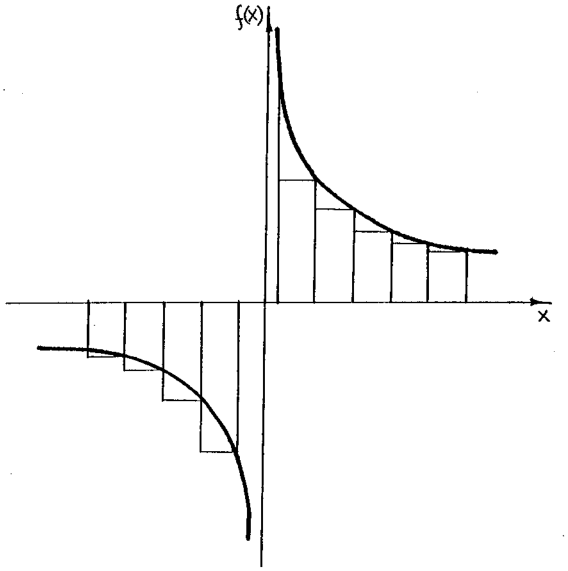


FIG. 4. Approximation of the integral of the function $f(x)=\text{sign}(x)|x|^{-1/4}$ by the method of rectangles.

$$\begin{aligned}\Delta\Phi=a_0\sum_{k=-N+1}^{N}\Bigg[&-\alpha\Theta_0\,\frac{\text{sign}(i_0+k\alpha\Theta_0)}{|i_0+k\alpha\Theta_0|^{1/4}}+\frac{4}{3}\,(|i_0\\&+k\alpha\Theta_0|^{3/4}-|i_0+(k-1)\alpha\Theta_0|^{3/4})\Bigg]+O(\alpha),\end{aligned} \tag{4.11}$$

where N is any integer $\sim 1/\alpha$ (the leading term in the asymptotic expression does not depend on the choice of N). The expression after the summation symbol in (4.11) has a simple geometric interpretation: apart from a constant factor, it is the difference between the area under the graph of the function $f(x)=\text{sign}(x)|x|^{-1/4}$ and under the approximating curve of the rectangle method with a step of $\alpha\Theta_0$ (Fig. 4). The sum of the terms in parentheses in (4.11) is $O(\alpha)$, since all these terms, other than the two extreme terms, cancel out.

If we now substitute $\xi=-i_0/\alpha\Theta_0$, we obtain

$$\begin{aligned}\Delta\Phi&=-\alpha^{3/4}a_0\Theta_0^{3/4}\sum_{k=-N+1}^{N}\frac{\text{sign}(-\xi+k)}{|-\xi+k|^{1/4}}+O(\alpha)\\&=-\alpha^{3/4}a_0\Theta_0^{3/4}\sum_{k=0}^{N-1}\left(\frac{1}{|-\xi+1+k|^{1/4}}-\frac{1}{|\xi+k|^{1/4}}\right)\\&\quad+O(\alpha).\end{aligned} \tag{4.12}$$

The sum in (4.12) can be expressed in the form of an integral. Indeed, from the definition of the Γ function

$$\Gamma\left(\frac{1}{4}\right)=p^{1/4}\int_0^{\infty}t^{-3/4}e^{-pt}\,dt,\quad p>0.$$

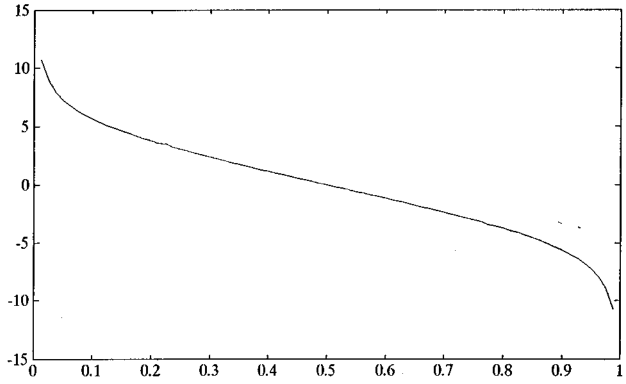


FIG. 5. Graph of the integral for $\Delta\Phi$ given by (4.13) as a function of ξ.

Hence for $\xi > 0$

$$\sum_{n=0}^{N-1} \frac{1}{(\xi+n)^{1/4}} = \frac{1}{\Gamma(1/4)} \sum_{n=0}^{N-1} \int_0^\infty t^{-3/4} e^{-(\xi+n)t}\,dt$$

$$= \frac{1}{\Gamma(1/4)} \int_0^\infty \frac{t^{-3/4}e^{-\xi t}}{1-e^{-t}}\,(1-e^{-Nt})dt.$$

Substituting this expression in (4.12), we find that

$$\Delta\Phi = \alpha^{3/4} a_0 \Theta_0^{3/4} \frac{1}{\Gamma(1/4)} \int_0^\infty \frac{t^{-3/4}(e^{-\xi t}-e^{-(1-\xi)t})}{1-e^{-t}} \times (1-e^{-Nt})dt + O(\alpha).$$

Passing to the limit as $N \to \infty$ in the integral, we finally obtain

$$\Delta\Phi = \alpha^{3/4} a_0 |\Theta_0|^{3/4} \frac{1}{\Gamma(1/4)} \int_0^\infty \frac{t^{-3/4}(e^{-\xi t}-e^{-(1-\xi)t})}{1-e^{-t}}\,dt + O(\alpha). \tag{4.13}$$

The modulus is used in (4.13) to ensure that this formula also gives the change in Φ for the transition from the $x>0$ to the $x<0$ hemisphere.

The integral in (4.13) is a function of ξ. Figure 5 shows a graph of this function. It has singularities of the form $\xi^{-1/4}$ and $(1-\xi)^{-1/4}$ at the points $\xi=0$ and $\xi=1$, respectively.

According to (4.4) and the definition of ξ, we have

$$c_1\alpha^{3/4} < \xi < 1 - c_1\alpha^{3/4}, \quad c_1 = \text{const} > 0. \tag{4.14}$$

When condition (4.14) is not satisfied, the expression given by (4.13) is in general invalid. However, it can be shown that, in this case too, if

$$c_2\alpha^3 < \xi < c_1\alpha^{3/4} \quad \text{or} \quad 1 - c_1\alpha^{3/4} < \xi < 1 - c_2\alpha^3, \tag{4.15}$$

then

$$\Delta\Phi = \alpha^{3/4} O[\xi^{-1/4} + (1-\xi)^{-1/4}].$$

The above analysis shows that Φ changes by only $O\{\alpha^{3/4}(\xi^{-1/4}+(1-\xi)^{-1/4})\}$ along the above streamline segment. Hence, in particular, $j_0 = j_* + O\{\alpha^{3/4}[\xi^{-1/4} + (1-\xi)^{-1/4}]\}$ where j_* is found from the equation $\Phi(j_*,0) = \Phi_-$. The quantity j_* is the value of j for which the trajectory of the averaged system with initial condition i_-, j_- crosses the separatrix. We can replace a_0 and Θ_0 in (4.13) with $a_* = a(j_*)$ and $\Theta_* = \Theta(j_*)$, respectively, without altering the order of magnitude of the remainder term.

The quantity ξ in (4.14) is a function of the initial data. A small change of order α in the initial values i_-, j_- then produces in general a large change of order 1 in ξ. Hence for small α it is best to treat ξ as a random variable. Following Refs. 11, 15, and 16, let us determine the distribution of this random variable. Let V_δ be a sphere of radius δ, centered on an initial point lying on a streamline. Assuming that each point of this sphere is an initial point of a streamline, let us calculate ξ for this line, using $V_{\delta,\alpha}^{(a,b)}$ to represent the part of the sphere V_δ for whose points the quantity ξ lies in the interval $(a,b) \in [0,1]$.

We shall put by definition that the probability for ξ to fall into the interval (a,b) is

$$P\{\xi \in (a,b)\} = \lim_{\delta\to 0} \lim_{\alpha\to 0} \frac{\mathrm{vol}(\mathbb{V}_{\delta,\alpha}^{(a,b)})}{\mathrm{vol}(\mathbb{V}_\delta)},$$

where the symbol vol represents the standard volume in $\mathfrak{R}^3$.

It can be shown that $P\{\xi \in (a,b)\} = b - a$ (the proof of this natural statement is laborious and will not be reproduced here). It follows that ξ can be treated as a random variable, distributed uniformly on the interval $[0,1]$. Correspondingly, $\Delta\Phi$ is also treated as a random variable whose distribution is given by (4.13). From (4.13) we find that the mean of $\Delta\Phi\alpha^{-3/4}$ is 0 in the leading approximation. The standard deviation of $\Delta\Phi\alpha^{-3/4}$ in the leading approximation is $a_*\Theta_*^{3/4}R/\Gamma(1/4)$ where $R \approx 4.5446$.

The formula for $\Delta\Phi$, given by (4.13), was checked numerically. The system (1.1) was integrated numerically with the help of program TraX[17,18] and the results are shown in Fig. 6 in the form of a graph of $\Delta\Phi(\alpha)$ for $j_0 = -1.6$, $\xi = 0.3$ (a) and $\xi = 0.8$ (b). The straight lines in Fig. 6 correspond to theoretical values of $\Delta\Phi$ given by (4.13) and the asterisks show values obtained by numerical integration of (1.1) for different values of α. These graphs demonstrate that there is good agreement between the theoretical formula (4.13) and the computer simulation of the system.

V. DIFFUSION OF THE ADIABATIC INVARIANT DUE TO MULTIPLE SEPARATRIX CROSSINGS

Motion in the averaged system on the i,j plane is periodic (Fig. 3). The function $\Phi(i,j)$ does not vary along the trajectory. The trajectory cuts the separatrix twice in one period. The behavior of the quantities i, j along a streamline of the original system can be described as follows. Well away from the separatrix, a point $i(t), j(t)$ follows [to within $O(\alpha)$] the trajectory of the averaged system. The function Φ undergoes a quasirandom jump of the order $\alpha^{3/4}$ [see (4.13)] as a small neighborhood of the separatrix is traversed. This means that, after the separatrix has been crossed, the point $i(t), j(t)$ follows a different trajectory of the averaged sys-

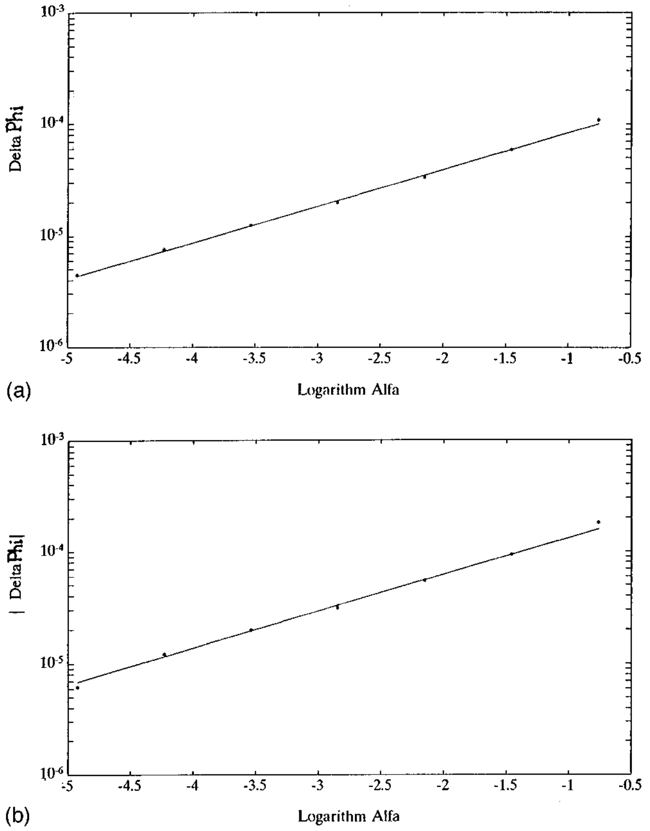


FIG. 6. Numerical verification of (4.13). The straight line corresponds to (4.13); asterisks show points obtained by numerical simulation: (a) $j_0=-1.6$, $\xi=0.3$, (b) $j_0=-1.6$, $\xi=0.8$.

tem. Subsequent crossings of the separatrix produce further jumps in Φ, and so on. Such kinds of behavior are sometimes referred to as intermittency.

Let us now examine the statistical properties of the jumps in Φ that accompany crossings of the separatrix. Suppose that successive crossings are characterized by values ξ_1 and ξ_2 of ξ (Sec. IV C). A small increase $\Delta\xi_1$ in ξ_1 leads to a change in the jump in Φ by $\sim\alpha^{3/4}\Delta\xi_1$. This additional change in Φ produces in the time $\sim 1/\alpha$ to the next crossing of the separatrix a change $\sim\alpha^{-1/4}\Delta\xi_1$ in the phase φ of the point (see Sec. III). Correspondingly, the quantity ξ_2 changes by the amount $\sim\alpha^{-1/4}\Delta\xi_1$, i.e., $d\xi_2/d\xi_1 \gg 1$. If we then adopt the "small causes, large effects" principle, we may naturally suppose that ξ_1 and ξ_2 are independent quantities, so that the variation of Φ that accompanies repeated crossing of the separatrix can be regarded as a random walk.

The change in Φ after N crossings of the separatrix is $\sqrt{N}\alpha^{3/4}$. For $N\sim\alpha^{-3/2}$ the resultant change in Φ is of order 1, which requires a time of order $\alpha^{-5/2}$ (i.e., the diffusion coefficient is $D\sim\alpha^{5/2}$). A point $i(t),j(t)$ will pass near a point i,j of the unit sphere in (x,y,z) space in a time of order $\alpha^{-5/2}$. Correspondingly, during this time, a streamline will pass close to any point of the unit sphere.

Figure 7 shows a section by the $y=0$ plane of a streamline that repeatedly crosses the separatrix; the figure clearly demonstrates the diffusion of the adiabatic invariant.

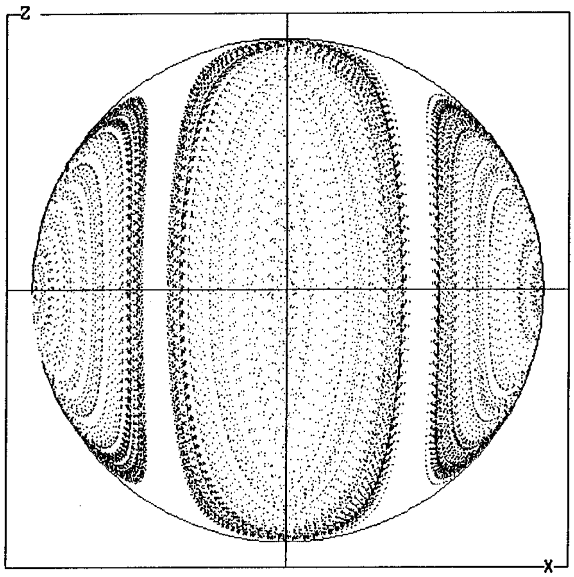


FIG. 7. Section of a streamline by the $y=0$ plane for repeated crossing of the separatrix.

The accumulation of jumps in the adiabatic invariant, which leads to chaotic motion in the above system, was demonstrated numerically in Ref. 6 and was referred to as *transadiabatic drift*. However, since the average of the leading term in $\Delta\Phi$ in (4.13) is 0, it is more natural to speak of *diffusion* rather than *drift*. The formula given by (4.13) explains this phenomenon and provides a numerical estimate for the chaos development time $(t\sim\alpha^{-5/2})$.

Let us finally estimate the maximum positive Lyapunov exponent λ for the above problem. After N crossings of the separatrix, the quantity $\Delta\xi_1$, introduced above, expands by a factor of $(\alpha^{-1/4})^N$ and this is accomplished in a time $t\sim N/\alpha$. Hence

$$\lambda\sim\frac{\ln(\alpha^{-1/4})^N}{N/\alpha}\sim\frac{1}{4}\,\alpha\,\ln\left(\frac{1}{\alpha}\right).$$

VI. CONCLUSION

We have shown that streamline chaos in a confined Stokes flow (1.1) arises as a result of the quasirandom changes in the adiabatic invariant of the system as the separatrix of the unperturbed flow is repeatedly crossed. Each trajectory of the perturbed system crosses the separatrix at time intervals of order $1/\alpha$ where α is the magnitude of the perturbation. This mechanism for the origin of chaos leads to the randomization of motion in the entire interior of the sphere for arbitrarily small perturbations, i.e., to the so-called strong chaos.

We have derived the basic formula for the change in the adiabatic invariant $\Delta\Phi$ that accompanies the crossing of the separatrix. It turns out that, when the separatrix is crossed, the adiabatic invariant undergoes a quasirandom change of the order of $\alpha^{3/4}$. We have used this formula as a basis for our estimate of the maximum positive Lyapunov exponent and the chaos development time.

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APPENDIX A: EVALUATION OF THE DERIVATIVES OF THE ADIABATIC INVARIANT

We shall now derive (3.4). We begin by considering the quantity $\Delta\Phi = \Phi(i+\Delta i, j) - \Phi(i,j)$ for $i>0$, $\Delta i>0$. We have

$$\Delta\Phi = -\int_{\Delta S} (\mathbf{v},\mathbf{n})\,d\sigma, \tag{A1}$$

where ΔS is part of the surface $\{J=j\}$ bounded by the unperturbed trajectories $\{I=i,\ J=j\}$ and $\{I=i+\Delta i,\ J=j\}$, $d\sigma$ is an area element on ΔS, and $\mathbf{n}$ is a unit normal on ΔS, chosen as indicated at the end of Sec. III. The minus sign in (A1) is due to the fact that the area on the surface $\{J=j\}$ that is bounded by the curve $\{I=i,\ J=j\}$ contracts with increasing i.

Let us now introduce the tangential vector $d\mathbf{l}$ on the perturbed trajectory $\{I=i,\ J=j\}$:

$$d\mathbf{l} = [\operatorname{grad} I, \operatorname{grad} J]\,dt, \tag{A2}$$

where dt is an element of time for motion along the trajectory. Next, we introduce the following vector on the curve $(I=i,\ J=j)$:

$$\Delta\mathbf{r} = \frac{[\operatorname{grad} J, \boldsymbol{\xi}]\Delta i}{(\operatorname{grad} J, \operatorname{grad} I, \boldsymbol{\xi})}, \tag{A3}$$

where $\boldsymbol{\xi}$ is an arbitrary vector that does not lie in the plane containing the vectors $\operatorname{grad} I$ and $\operatorname{grad} J$. The vector $\Delta\mathbf{r}$ is orthogonal to $\operatorname{grad} J$ and its end point lies on the curve $\{I=i+\Delta i,\ J=j\}$ to within $O(\Delta i^2)$ since $(\operatorname{grad} J, \Delta\mathbf{r}) = \Delta i$. Hence, using (A1)–(A3), we obtain the following expression in the leading approximation in Δi:

$$\Delta\Phi = \oint (\mathbf{v}, [\Delta\mathbf{r}, d\mathbf{l}]) = -\oint (\mathbf{v}, \operatorname{grad} J)\Delta i\ d\mathbf{l}.$$

Hence, and from (3.2), we obtain the second formula in (3.4) for $i>0$. This formula in the case $i<0$ and the first formula in (3.4) can be proved similarly.

We shall now show, without resorting to detailed calculation, that $\Phi(i,j)$ is an integral of the averaged system. Let $I=i(\tau)$, $J=j(\tau)$, $\tau=\alpha t$ be a solution for the averaged system and consider the shift of the contour $\{I=i(0),\ J=j(0)\}$ along the trajectory of the perturbed system in a time τ/α. The flux of the vector $\mathbf{v}$ through a surface spanned on the shifted contour denoted by L, coincides with the flux through a surface spanned on the original contour, i.e., with $\Phi(i(0),j(0))$, because the flow is incompressible. On the other hand, L lies in the $O(\alpha)$ neighborhood of the contour $\{I=i(\tau),\ J=j(\tau)\}$ according to the theorem on the accuracy of the averaging method.[8,13] Hence the flux of the vector $\mathbf{v}$ through a surface spanned on L is $\Phi(i(\tau),j(\tau))+O(\alpha)$. Equating these two expressions for the flux, and passing to the limit as $\alpha\to 0$, we obtain $\Phi(i(\tau),j(\tau)) = \Phi(i(0),j(0))$, which was required.

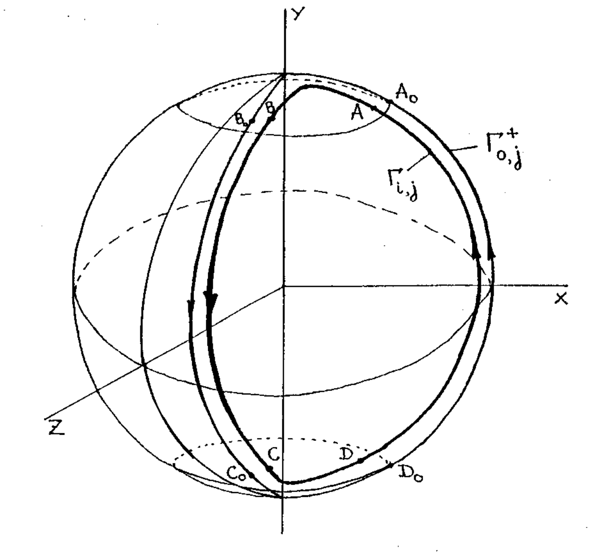

FIG. 8. Schematic representation of the streamlines $\Gamma^+_{0,j}$, $\Gamma_{i,j}$ of unperturbed flow.

APPENDIX B: ASYMPTOTIC EXPANSION FOR THE ADIABATIC INVARIANT NEAR THE SEPARATRIX

In this appendix, we derive (4.1) and (4.3). To be specific, we consider the case $i>0$. According to (3.6), we have

$$\Phi(i,j) = -\frac{1}{2}\,j\oint_{\Gamma_{i,j}} z^3\,dy, \quad \Phi_0(j) = -\frac{1}{2}\,j\oint_{\Gamma_j} z^3\,dy. \tag{B1}$$

Let us choose a particular δ, $0<\delta<1$ and consider the planes $y=1-\delta$ and $y=-1+\delta$. For sufficiently small i, they cut the contour $\Gamma_{i,j}$ and divide it into four parts (Fig. 8): segment BC near the $x=0$ plane, segment DA near the $z=0$ plane, and two other segments, AB and CD, near the poles of the sphere. Similarly the contour Γ_0^+ is divided into four segments by points A_0, B_0, C_0, D_0 (Fig. 8).

According to (2.1), we have along $\Gamma_{i,j}$

$$i = xz^4, \quad jz^3 = x^2+y^2+z^2-1. \tag{B2}$$

Hence we find that on BC

$$y^2 = jz^3 - z^2 + 1 + O_\delta(i^2).$$

The subscript δ of the symbol $O_\delta(i^2)$ is necessary because this estimate is not, generally speaking, uniform in δ. We now have for the segment BC

$$\Phi^{\mathrm{BC}}(i,j)=-\frac{1}{2}\,j\int_{\mathrm{BC}}z^3\,dy=-\frac{1}{2}\,j\int_{\mathrm{B_0C_0}}z^3\,dy+O_\delta(i^2)$$

$$=\Phi_0(j)+f_1(\delta)+O_\delta(i^2),$$

where $f_1(\delta)=O(\delta^{5/2})$. We note that the function $f_1(\delta)$ does not depend on i.

Again, it follows from (B2) that on DA

$$x=\sqrt{1-y^2}+O_\delta(i^{1/2}).$$

Hence for the segment DA

$$\Phi^{\mathrm{DA}}(i,j)=-\frac{1}{2}\,j\int_{\mathrm{DA}}z^3\,dy=-\frac{1}{2}\,j\int_{\mathrm{DA}}\left(\frac{i}{x}\right)^{3/4}dy$$

$$=-\frac{1}{2}\,ji^{3/4}\int_{\mathrm{DA}}\frac{dy}{(1-y^2)^{3/8}}+O_\delta(i^{5/4})$$

$$=-\frac{1}{2}\,ji^{3/4}\int_{-1}^{1}\frac{dy}{(1-y^2)^{3/8}}+i^{3/4}f_2(\delta)$$

$$+O_\delta(i^{5/4}),$$

where $f_2(\delta)=O(\delta^{5/8})$.

Evaluating the integral, we obtain[19]

$$\Phi^{\mathrm{DA}}(i,j)=-ji^{3/4}\,\frac{\sqrt{\pi}}{2}\,\frac{\Gamma(5/8)}{\Gamma(9/8)}+i^{3/4}f_2(\delta)+O_\delta(i^{5/4}).$$

When we consider the integral on the segment AB, it is convenient to transform to integration with respect to z:

$$\Phi^{\mathrm{AB}}(i,j)=-\frac{1}{2}\,j\int_{\mathrm{AB}}z^3\,dy$$

$$=-\frac{1}{2}\,j\int_{z_{\mathrm{A}}}^{z_{\mathrm{B}}}\frac{z^3(3jz^2-2z+8i^2/z^9)}{[1-(z^2-jz^3+i^2/z^8)]^{1/2}}\,dz,$$

where $z_{\mathrm{A}}\sim\sqrt{\delta}$, $z_{\mathrm{B}}\sim i^{1/4}/\delta^{1/8}$. We have

$$\Phi^{\mathrm{AB}}(i,j)=-\frac{1}{2}\,j\int_{z_{\mathrm{A_0}}}^{0}\frac{z^3(3jz^2-2z)}{(1-z^2+jz^3)^{1/2}}\,dz+i^{3/4}O(\delta^{5/8})$$

$$+O_\delta(i^{5/4})=i^{3/4}f_3(\delta)+f_4(\delta)+O_\delta(i^{5/4}),$$

where $f_3(\delta)=O(\delta^{5/8})$, $f_4(\delta)=O(\delta^{5/2})$.

Combining the above expressions, we obtain

$$\Phi(i,j)=\Phi_0(j)+f_5(\delta)+i^{3/4}(\tfrac{4}{3}\,a+f_6(\delta))+O_\delta(i^{5/4}),$$

$$a=-\frac{3\sqrt{\pi}}{8}\,j\,\frac{\Gamma(5/8)}{\Gamma(9/8)},\tag{B3}$$

where $f_5(\delta)=O(\delta^{5/2})$, $f_6(\delta)=O(\delta^{5/8})$.

Since the left-hand side is independent of δ, we find from (B3) that

$$\Phi(i,j)=\Phi_0(j)+\tfrac{4}{3}ai^{3/4}+O(i^{5/4}).\tag{B4}$$

For $\Theta=\partial\Phi_0/\partial j$ we find from (4.2) that

$$\Theta=\frac{1}{2}\oint_{\Gamma_j}z^5\,dt=\frac{1}{2}\oint_{\Gamma_j}\frac{z^4}{y}\,dz=\int_0^{z_m}\frac{z^4\,dz}{\sqrt{jz^3-z^2+1}},$$

where z_m is the maximum absolute value of z on Γ_j.

The coefficient a in (B4) can also be found in another way:

$$a=\lim_{i\to 0}i^{1/4}\,\frac{\partial\Phi}{\partial i}$$

$$=-\lim_{i\to 0}i^{1/4}\oint_{\Gamma_{i,j}}(\mathrm{grad}\,I,\mathbf{v})dt$$

$$=-\lim_{i\to +0}i^{1/4}\oint_{\Gamma_{i,j}}\frac{3x(x^2+y^2+z^2-1)}{z^4}\,dt$$

$$=-3\lim_{i\to +0}i^{1/4}j\oint_{\Gamma_{i,j}}\frac{x}{z}\,dt$$

$$=-3j\lim_{i\to +0}\oint_{\Gamma_{i,j}}x^{5/4}\,dt$$

$$=-3j\int_{S^+}x^{5/4}\,dt$$

$$=-3j\int_{S^+}\frac{x^{5/4}}{-8xy}\,dt$$

$$=-\frac{3}{4}\,j\int_{S^+}\frac{x^{1/4}}{(1-x)^{1/2}}\,dt$$

$$=-\frac{3\sqrt{\pi}}{8}\,j\,\frac{\Gamma(5/8)}{\Gamma(9/8)}.$$

APPENDIX C: ON THE ACCURACY OF THE FORMULA FOR THE CHANGE IN THE ADIABATIC INVARIANT

We shall now estimate the residual term in the expression for the change in the adiabatic invariant, given by (4.11).

We have

$$i_k-i_{k-1}=\int_{\gamma_k}\dot{I}\,dt,\quad j_k-j_{k-1}=\int_{\gamma_k}\dot{J}\,dt.\tag{C1}$$

Replacing the integrals in (C1) with integrals along the trajectories of the unperturbed problem, estimating the error introduced as a result of this, and using (3.2) and (3.4), we obtain

$$\begin{aligned}i_k-i_{k-1}&=\delta I_k+O(\alpha^2),\\ j_k-j_{k-1}&=\delta J_k+\alpha^2O(|i_k|^{-1/2}),\end{aligned}\tag{C2}$$

where

$$\delta I_k=\alpha\,\frac{\partial\Phi(i_k,j_{k-1})}{\partial j},\quad \delta J_k=-\alpha\,\frac{\partial\Phi(i_k,j_{k-1})}{\partial i}.\tag{C3}$$

The positioning of the subscripts $k-1$ and k on the right-hand sides of (C3) represents the fact that, along γ_k, the main change in I occurs on the segment $M_{k-1}\tilde{M}_k$, and the main change in J occurs on $\tilde{M}_k M_k$.

From (C2), (C3), and (4.1), we obtain

$$\begin{aligned} i_k-i_{k-1}&=\alpha\Theta(j_{k-1})+\alpha O(|i_k|^{3/4})+O(\alpha^2),\\ j_k-j_{k-1}&=-\alpha a_0|i_k|^{-1/4}\operatorname{sign}(i_k)+\alpha O(|i_k|^{1/4})\\ &\quad+\alpha^2 O(|i_k|^{-1/2}). \end{aligned} \tag{C4}$$

To begin with, consider the turns of γ_k for which $k\neq 0,1,2$. We then have $i_k>c_3^{-1}\alpha$ where $c_3=\text{const}>0$. Hence terms of order α in (C4) dominate over terms of order α^2. From (C4) we find that

$$\begin{aligned} &i_k-i_{k-1}>c_4^{-1}\alpha,\quad c_4=\text{const}>0,\\ &i_k=i_0+k\alpha\Theta(j_0)+O(|i_k|^{7/4}),\\ &j_k=j_0+O(|i_k|^{7/4}). \end{aligned} \tag{C5}$$

From (C2), (C3), and (4.1) we also obtain the formula for the change in the adiabatic invariant per turn of γ_k:

$$\begin{aligned} \Delta\Phi_k&=\Phi_k-\Phi_{k-1}\\ &=\Phi(i_{k-1}+\delta I_k,j_{k-1}+\delta J_k)-\Phi(i_{k-1},j_{k-1})\\ &\quad+\alpha^2 O(|i_k|^{-1/2}). \end{aligned} \tag{C6}$$

We note that the right-hand side of this expression, when it is formally expanded into powers of α, is a quantity of the order of α^2 (as should be the case in view of the adiabatic invariance of Φ).

The formula given by (C6) can be transformed with the help of (C3) and (C4):

$$\begin{aligned} \Delta\Phi_k&=\Phi\left(i_{k-1}+\alpha\,\frac{\partial\Phi(i_k,j_{k-1})}{\partial j},j_{k-1}\right)-\Phi(i_{k-1},j_{k-1})\\ &\quad-\alpha\,\frac{\partial\Phi(i_k,j_{k-1})}{\partial j}\,\frac{\partial\Phi(i_k,j_{k-1})}{\partial i}+\alpha^2 O(|i_k|^{-1/2}). \end{aligned}$$

Replacing j_k on the right-hand side of this expression with j_0, and estimating the error introduced thereby with the help of (C5), we obtain

$$\begin{aligned} \Delta\Phi_k&=\Phi\left(i_{k-1}+\alpha\,\frac{\partial\Phi(i_k,j_0)}{\partial j},j_0\right)-\Phi(i_{k-1},j_0)\\ &\quad-\alpha\,\frac{\partial\Phi(i_k,j_0)}{\partial j}\,\frac{\partial\Phi(i_k,j_0)}{\partial i}+\alpha^2 O(|i_k|^{-1/2}). \end{aligned}$$

We now put

$$\begin{aligned} &\Phi=\Phi_a(i,j)+\Phi_b(i,j),\\ &\Phi_a(i,j)=\Phi_0(j)+\tfrac{4}{3}a(j)|i|^{3/4}. \end{aligned}$$

According to (4.1), $\Phi_b(i,j)=O(|i|^{5/4})$. In terms of the new notation:

$$\begin{aligned} \Delta\Phi_k&=\Phi_a\left(i_{k-1}+\alpha\,\frac{\partial\Phi(i_k,j_0)}{\partial j},j_0\right)-\Phi_a(i_{k-1},j_0)\\ &\quad-\alpha\,\frac{\partial\Phi(i_k,j_0)}{\partial j}\,\frac{\partial\Phi_a(i_k,j_0)}{\partial i}\\ &\quad+\Phi_b\left(i_{k-1}+\alpha\,\frac{\partial\Phi(i_k,j_0)}{\partial j},j_0\right)-\Phi_b(i_{k-1},j_0)\\ &\quad-\alpha\,\frac{\partial\Phi(i_k,j_0)}{\partial j}\,\frac{\partial\Phi_b(i_k,j_0)}{\partial i}+\alpha^2 O(|i_k|^{-1/2}). \end{aligned}$$

The set of terms containing Φ_b explicitly amounts to $\alpha^2 O(|i_k|^{-3/4})$. Hence

$$\begin{aligned} \Delta\Phi_k&=\Phi_a\left(i_{k-1}+\alpha\,\frac{\partial\Phi(i_k,j_0)}{\partial j},j_0\right)-\Phi_a(i_{k-1},j_0)\\ &\quad-\alpha\,\frac{\partial\Phi(i_k,j_0)}{\partial j}\,\frac{\partial\Phi_a(i_k,j_0)}{\partial i}+\alpha^2 O(|i_k|^{-3/4}). \end{aligned}$$

In the last expression we now replace $\partial\Phi(i_k,i_0)/\partial j$ with Θ_0, i_k with $i_{k-1}+\alpha\Theta_0$, and i_{k-1} with $i_0+(k-1)\alpha\Theta_0$; this introduces an additional error $\alpha^2 O(|i|^{-1/2})$. Hence

$$\begin{aligned} \Delta\Phi_k&=a_0\left[\left(\frac{4}{3}\,|i_0+k\alpha\Theta_0|^{3/4}-\frac{4}{3}\,|i_0+(k-1)\alpha\Theta_0|^{3/4}\right)\right.\\ &\quad\left.-\alpha\Theta_0\,\frac{\operatorname{sign}(i_0+k\alpha\Theta_0)}{|i_0+k\alpha\Theta_0|^{1/4}}\right]+\alpha^2 O(|i_k|^{-3/4}). \end{aligned} \tag{C7}$$

We recall that this relation was obtained for $k\neq 0,1,2$. For $k=0,1,2$, the problem becomes simpler because we can then use the expansions obtained from (4.1), (4.4), and (C4):

$$\begin{aligned} &\Phi(i,j)=\Theta_0(j-j_0)+\tfrac{4}{3}a_0|i|^{3/4}+O(\alpha),\\ &i_k-i_{k-1}=\alpha\Theta_0+O(\alpha^{25/16}),\\ &j_k-j_{k-1}=-\alpha a_0|i_k|^{-1/4}\operatorname{sign}(i_k)+O(\alpha). \end{aligned}$$

Hence we find that $\delta\Phi_k$ is equal to the leading term in (C7) with an error of $O(\alpha)$.

Summing the above expressions for $\Delta\Phi_k$, and recalling that

$$\alpha\sum_{k=-N+1}^{N}|i_k|^{-3/4}=O(1),\quad k\neq 0,1,2,$$

we obtain the estimate given by (4.11).

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