Cosmic lacunarity

A. Provenzale
Istituto di Cosmogeofisica, Corso Fiume 4, I-10133 Turin, Italy

E. A. Spiegel
Department of Astronomy, Columbia University, New York, New York 10027

R. Thieberger
Department of Physics, Ben Gurion University, Beer Sheva, Israel

(Received 15 July 1996; accepted for publication 20 September 1996)

The present distribution of galaxies in space is a remnant of their formation and interaction. On a large enough scale, we may represent the galaxies as a set of points and quantify the structures in this set by its generalized dimensions [Beck and Schlögl, *Thermodynamics of Chaotic Systems* (Cambridge University Press, Cambridge, 1986); Paladin and Vulpiani, Phys. Rep. **156**, 147 (1987)]. The results of such evaluation are often taken to be evidence of a fractal (or multifractal) distribution of galaxies. However, those results, for some scales, may also reveal the presence of singularities formed in the gravitational processes that produce structure in the galaxy distribution. To try to make some decision about this issue, we look for the more subtle galactic lacunarity. We believe that this quantity is discernible in the currently available data and that it provides important evidence on the galaxy formation process. *© 1997 American Institute of Physics.* [S1054-1500(97)00701-5]

The luminous matter that makes up the galaxies constitutes but a few percent of the total mass of the universe.[1] The visible galaxies are thus like whitecaps whose motion is a clue to what the darker, deeper seawater is doing. And, just as the whitecaps form in the most singular parts of the water surface, the galaxies must represent accumulations of luminous matter in potential wells created by the dark matter in a phenomenon called bias. Simulations of such gravitational processes indeed reveal the kind of inhomogeneous distribution of matter suggested by the observed galaxy distribution, though there is as yet no general agreement on the proper way to model the influence of the invisible matter. We do not enter into that discussion here but concern ourselves instead with the analysis of the observed distribution of galaxies treated as an abstract point set. We raise the question of what the optimal description of this set is and examine its fractal properties. (We use the word fractal in the original generalized sense of which multifractals are a particular case.) The presently available data of the Center for Astrophysics redshift catalog[2–4] permit reasonable estimates of a few of the Renyi dimensions[5–7] and, remarkably, allow the detection of the corresponding lacunarities.[8]

I. COSMOLOGICAL BACKGROUND

For those readers who have thought more about point sets than cosmology, we begin with some background on the nature of the cosmological flow. This can be treated quite simply since the processes that we are concerned with had their main development after the recombination era, when the universe had cooled sufficiently so that protons and electrons combined to form neutral hydrogen. This event took place one hundred thousand years after the beginning of the presently observed global expansion that may adequately, for our purposes, be discussed in the context of Newtonian physics. We need only consider the nearby portions of the universe (in space-time) for our study and for this we need only the following elementary notions.

The simplest models assume spatial isotropy and, since all points are equivalent, any point can be used as the origin of coordinates. Our own location will then serve this purpose. Consider a galaxy moving with the general expansion. If it is at distance d_0 from us at time t_0, it is at distance $d=a(t)d_0$ at time t, where $a(t)$ is called the scale factor and $a(t_0)=1$. Considerations of gravitational dynamics together with the assumption of isotropy of the local universe at large give simple differential equations for $a(t)$. All that matters for our discussion is that $a(t)$ is presently increasing, though there are interesting issues about its long time behavior that are not at present resolved.

In discussing the motion of an individual galaxy, it is convenient to use a coordinate system that shares the general expansion given by $a(t)$. If a galaxy is at position $\mathbf{r}(t)$ we may introduce a new coordinate $\mathbf{x}$ such that $\mathbf{r}=a(t)\mathbf{x}$. Then the velocity of the galaxy is $\dot{\mathbf{r}}=a(\dot{\mathbf{x}}+H\mathbf{x})$ where $H=\dot{a}/a$ and $H_0=H(t_0)$ is called the Hubble constant if t_0 is the present time. For a galaxy at rest with respect to the mean expansion ($\dot{\mathbf{x}}=\mathbf{0}$) we have $\dot{\mathbf{r}}=H\mathbf{r}$, which, for relatively small distances and recent times, gives Hubble's law, $|\dot{\mathbf{r}}|=H_0|\mathbf{r}|$. Clearly, H_0^{-1} is a rough indicator of the time since the expansion began and this is presently thought to be a few times 10^{10} years.

In general, galaxies are not frozen into the global expansion (called the Hubble flow), but move with respect to the background flow; these relative motions are called peculiar velocities. There is also an acceleration given by

$$\ddot{\mathbf{r}} = a\left(\ddot{\mathbf{x}} + 2H\dot{\mathbf{x}} + \frac{\ddot{a}}{a}\mathbf{x}\right). \tag{1}$$

We see that, in the expanding coordinate system, there are fictitious forces analogous to the Coriolis and centrifugal forces in rotating frames. In particular, the force $-H(t)\dot{\mathbf{x}}$ represents a cosmic drag that tends to bring all galaxies to rest with respect to the mean expansion. Peculiar velocities result from some combination of this effect, initial conditions and gravitational interactions.

A photon reaching us from a distant galaxy will suffer the analogue of the cosmic drag, more aptly understood as the effect of the expansion on its wavelength. This effect, called the cosmologic redshift, is taken as evidence for a universal expansion so that, for a galaxy with zero peculiar velocity, the redshift indicates the distance to the galaxy. Over many decades, it has been possible, through a combination of ingenious techniques, to estimate the Hubble constant and this provides a means of gauging the distance to a galaxy once its redshift has been measured.[1,9] However, there are difficulties in such determination.

One problem is that the peculiar velocities also contribute to the redshift and this introduces some uncertainty into the distance estimates based on redshifts. Since the relative importance of this effect generally decreases with distance and increased cosmic redshifts, this difficulty may one day become less important as the distances to which redshift observations are made are increasing. But there are also systematic errors in clusters whose member galaxies interact to induce ordered local fluctuations in the redshifts. Some investigators try to remove such effects from the data by various devices, but we prefer to deal for now with the unadorned data, leaving such revisions for possible future study.

In the conventional usage, the redshift, denoted as z, is the fractional change in the wavelength of a photon reaching us from its galaxy of origin. The Hubble constant, which has the dimension of inverse time, is measured in units of km/s/Mpc, where the Mpc (or Megaparsec) is 3×10^{24} cm. Frequently, one introduces a scaled Hubble constant $h = H_0/\hat{H}_0$ where $\hat{H}_0 = 100$ km/s/Mpc. At present, h is thought to lie between 0.5 and 0.8, so that a value of 3/4 is a reasonable number to use.[10] In these terms, we represent the redshift in distance units as $cz/(h\hat{H}_0)$ where $c = 3\times10^5$ km/s is the speed of light. We shall generally not worry about the factor h^{-1} in this discussion. We stress that though we express redshifts as equivalent distances, all of our present study is confined to a space in which the radial coordinate is a measured redshift.

The redshifts of galaxies have been determined for a reasonably homogeneous sample of some thirty thousand galaxies and catalogued in the CfA-ZCAT (Center for Astrophysics, Harvard, Catalogue).[2–4] For each of these galaxies, the angular position of the galaxy on the sky is also known, as it is for some million other galaxies. Indeed, until recent years, studies on the distribution of the galaxies dealt with this projection on the celestial sphere of the galactic positions. For our purposes then, we can assign spherical coordinates $\mathbf{X} = (cz/\hat{H}_0, b, \ell)$, to each galaxy in the redshift catalogue, where the angular coordinates b and ℓ are called Galactic latitude and longitude. Figure 1 shows the projection of the galaxy positions from the catalogue onto the Galactic plane, $b = 0$. The streaks in the figure are probably artifacts caused by the internal motions of clusters of galaxies that modify the observed redshifts.

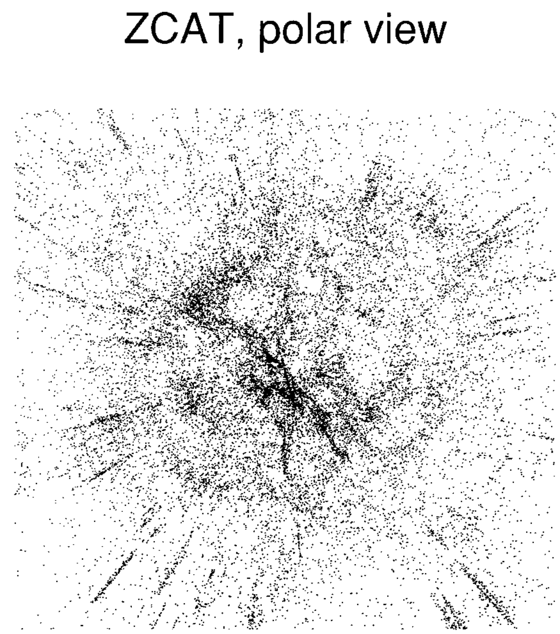

FIG. 1. Projection on the plane of our Galaxy of the CfA compilation of the ZCAT. We show only galaxies having galactic latitude larger than 10^0 N or smaller than -10^0 S. The observer is placed at the center of the field shown and the size of the box is approximately 240 Mpc.

II. GENERALIZED DIMENSIONS

In computing the generalized dimensions for the galaxy distribution, it is desirable to slightly modify the standard formulae for the generalized correlation integrals. In this section we explain why this is so and what we need to do. Since the matter is simple, we can keep the discussion on an intuitive and nontechnical level.

Suppose we have N galaxies in our sample and that we indicate the position of the ith galaxy by $\mathbf{X}_i$. We may count $\mathcal{N}_i(r)$, the number of galaxies lying within distance r of the ith galaxy. (Since the data are available for only relatively nearby galaxies, we need not worry about travel time of light.) We may also average this quantity over the galaxies. However, since the sample occupies a certain volume, some galaxies are rather near the edge of the volume and we would like to exclude them from this averaging. So we compute $\mathcal{N}(r) = M^{-1}\Sigma_{i=1}{}^{M}\mathcal{N}_i(r)$ where the average is over a subset $\mathcal{S}$ of $M < N$ galaxies chosen away from the edge (in redshift space) of the sample. In practice, we have experimented with the definition of this set and have found that it possible to choose it so that the average is not sensitive to details of the choice. Apart from the fact that $\mathcal{S}$ is typically a proper subset of the galaxies, $\mathcal{N}(r)/N$ is the usual correlation integral $C_2(r)$ and, for small r, we write $C_2(r) \propto r^{D_2}$, in the standard way.[11]

In the astrophysical literature, the practice of coping with the finiteness of the set of galaxies is not uniform and some authors prefer to keep all the galaxies in the observed sample and introduce a fictitious and randomly distributed set of background points to deal with the problem imposed by boundaries.

Another variation in the procedures found in the literature is that many authors work with the correlation. To compare this with the C_2 used in fractal studies, we write $N(r) = 4\pi\int_0^r n(s)s^2 ds$. (Some readers may prefer to think of the appropriate Lebesgue integral here.) For a random distribution, we would have $n(r) = n_0$, where n_0 is some suitable constant and, in that case, $N(r) \propto r^3$. With this case as a reference, one may write $n(r) = n_0[1+\xi(r)]$ where ξ is known as the correlation function. When ξ is greater than zero, the galaxies are said to be correlated.

All this is straightforward but a certain confusion has arisen because some people look for scaling in $\mathcal{N}$ (or $1+\xi$) while others look for it in ξ. This presents no problems when $\xi(r) \propto r^{-\gamma}$ (with $\gamma > 0$) and we consider sufficiently small r. For then we have $D_2 + \gamma = 3$ and the two approaches are equivalent. But the shortage of data sometimes forces one to include somewhat larger r than is suitable for recovering this result and therefore there is some disagreement about the scaling exponents for the galaxies. For reasonably nearby galaxies, early workers found $\gamma = 1.8$ and this result is still widely adopted.[1,9] For our present purposes, the exact value is not so important as the question of whether there really is scaling.

We may then state the (slightly modified) formula for the qth moment of the galactic point set as

$$C_q(r) = \left(\frac{1}{M}\Sigma_i\left[\frac{1}{N}\Sigma_j\Theta(r - |\mathbf{X}_i - \mathbf{X}_j|)\right]^{q-1}\right)^{1/(q-1)}, \quad (2)$$

where the summation on j is over the points in the set whose coordinates in our space of three dimensions are denoted by $\mathbf{X}_j$, the summation over i is over the subset $\mathcal{S}$ and Θ is the Heaviside function. Here we restrict ourselves to cases with positive integer values of q.

We expect that for small r

$$\log C_q = D_q \log\frac{r}{r_0} + \dots, \quad (3)$$

where D_q is the generalized or Renyi dimension and we have introduced a characteristic separation, r_0. The dots represent the remaining terms in this expansion in powers of log r and the next term is typically of order unity. We should also be prepared for the possibility that this simple behavior holds only when r lies in some finite range of scales.

III. OBSERVATIONAL LOCAL DIMENSIONS

The task of measuring redshifts is difficult, especially as one goes to ever farther, hence fainter, galaxies. Nevertheless, the number of known redshifts is growing rapidly and the application of the scaling (or fractal) analysis methods will be useful in analyzing the data. As yet, we believe that it is not yet clear whether the distribution of galaxies is really fractal. In this section we describe the dimensions of this distribution using the three-dimensional data of the CfA Catalog[2] of redshifts, which is a compilation from several sources.

There are almost no galaxies observed in the neighborhood of the plane of our Galaxy because of the obscuration by interstellar matter. The data are therefore conveniently divided into two portions, representing the northern ($b > 10°$) and southern ($b < -10°$) galactic hemispheres. There are also several, rather larger, two-dimensional data sets giving just the angular positions of the galaxies. Along with several others, we have studied the dimensions of these various point sets, but here we shall describe the most interesting one, the northern hemisphere galaxies in the CfA Catalogue. In that sample, there are about 15000 galaxies and we use a central set $\mathcal{S}$ (see the previous section) with $M \approx 5000$ galaxies.

In a situation of this kind, one expects to find scaling behavior, if there is any at all, only over a limited range of values of r. The range of scaling behavior may be seen in plots of log C_q against log r. In the study of the galaxy distribution, an effective way to bring out the limitation of the range of scales is to put a least-square-fit line through each set of three successive points in the log C_q–log r plot and calculate its slope. This slope is called $D_q(r)$, where r is the value of r for the middle point. In the plot of $D_q(r)$ against r, any range in which D_q is reasonably constant is said to show scaling behavior.

Now we turn to the analysis of the northern ZCAT, avoiding, as stated, the region where the obscuration from galactic dust is significant. A more serious problem arises because the extent of the catalogue is limited by the faintness to which galaxies may be observed. Galaxies may be seen as faint either because they are rather distant or because they are intrinsically faint. If care is not taken, an observed sample, limited according to faintness, may not select galaxies in an appropriate volume. To minimize any spurious dependence of the density of galaxies on distance from the observer arising from this selection effect, we select only that portion of the ZCAT with 2 Mpc $\leqslant zc/\hat{H}_0 \leqslant$ 120 Mpc.

In Fig. 2 we show results of the scaling analysis of this portion of the ZCAT. The error bars shown represent the 3 σ errors from fifty different choices of the galaxies used as centers to compute the correlation integral. We see that the results show ranges of r with nearly constant $D_q(r)$ for moments of order larger than three, with $D_q \approx 1.5$ on scales from 0.7 up to 15 h^{-1} Mpc. It is significant that, for these data, no scaling is observed for $q = 2$. Also noteworthy is the lack of scaling for r larger than about 20 h^{-1} Mpc even for the larger q.

It seems plausible that the behavior at the larger values of r, with $D_q(r)$ increasing with r, may represent an approach to the limiting value of 3. At the very largest scales, the gravitational dynamics that is central to the formation of

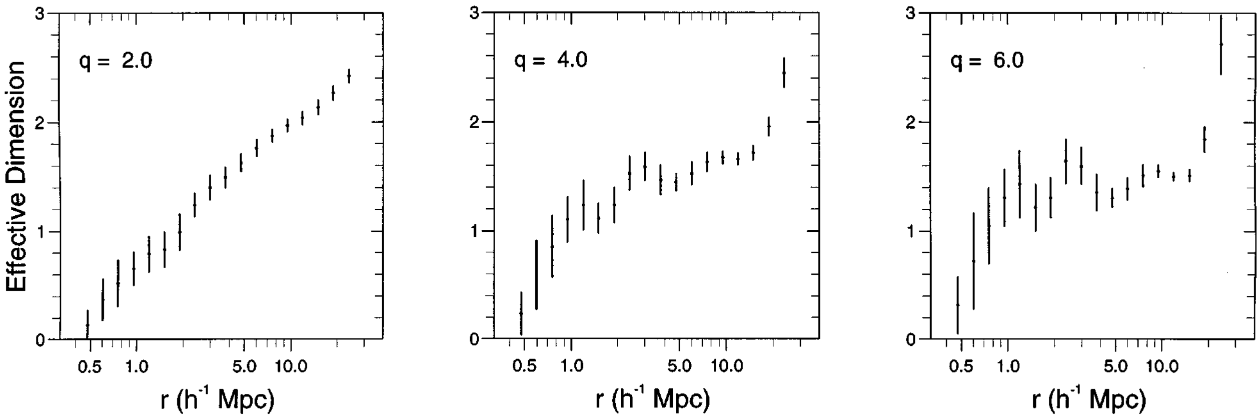

FIG. 2. Results of the scaling analysis of the northern portion of the ZCAT. We show $D_q(r)$ for $q=2,4,6$.

structures would not yet have had sufficient time to produce observable effects. In this interpretation, very large structures have not yet condensed out and so the $D_q(r)$ ought to be three for large r as for an uncorrelated distribution (see, however, Coleman and Pietronero[12] for a different view). For intermediate scales, on the other hand, it is supposed that local inhomogeneities have collapsed, most rapidly along their smallest diameters, to form flat structures called Zel'dovich pancakes.[13] On the smallest scales (a few Mpc[1,9]), other causes of structure have been postulated, including cosmic strings,[14] fractal cascades[12,15–23] and spherical collapse.[24–27]

IV. THE SINGULARITY PICTURE

A number of attempts to understand the galaxy distribution are based on simulations of the dynamics of a self-gravitating collisionless fluid. The initial conditions adopted are intended to reflect earlier events concocted to make the outcome look like the present universe. Many of them do produce a complex skein of cosmic crêpes, but these have not had enough time to evolve very much and it is not clear whether there has been adequate opportunity to develop fractal structures. On the other hand, most of the simulations do lead to the production of a number of strong local concentrations of mass[24–28] and these point to the possibility that the local density concentrations, manifested as clusters of galaxies, may be singular density distributions. We next describe a recent attempt[29] to see whether on the smallest observed scales a distribution of density singularities may represent the observed $D_q(r)$ as faithfully as a conventional fractal picture (though it must be admitted that neither is as yet fully satisfactory).

Consider a local density distribution of the form $n(R) \propto (R/R_0)^{-\alpha}$ where R is the distance from the singularity. If we suppose that α is between 1.8 and 2, we find that the scaling behavior of a random superposition of such singularities provides a fair description of the scaling found in the galaxy distribution. Already in the simplest case of a *single* condensation of this type we see that, for large positive q, the correlation integrals are dominated by the contributions from the regions of the distribution near to the singularity. The number of galaxies within distance r of any galaxy very near to the singularity is $\mathcal{N}_s(r) \propto r^{3-\alpha}$. For any finite value of q, to get the actual value of C_q, we need to do some averaging and the value obtained for D_q will depend weakly on the behavior of $\mathcal{N}$ for galaxies away from the singularity. From such reasoning in the general case we see that the expression for $C_q(r)$ tends to $\mathcal{N}_s(r)$ for $q \to \infty$.

For a singular density distribution with $n(R) \to (R/R_0)^{-\alpha}$ for small R, the scaling behavior is such that $D_\infty = 3-\alpha$, in analogy with the more familiar formula $D_2 = 3-\gamma$. For any finite q, the value of the dimension D_q is slightly larger than $3-\alpha$, because of the ''dressing'' of the dimension by the contribution of points away from the singularity. For an isolated singularity, it is possible to determine analytically the scaling behavior of the correlation function $\xi(r)$, and, hence, the value of D_2 at small scales. This is given by $D_2 = 6-2\alpha$. For sufficiently strong singularities ($\alpha > 1.5$), one thus finds approximate scaling behavior for any finite value of $q \geqslant 2$.

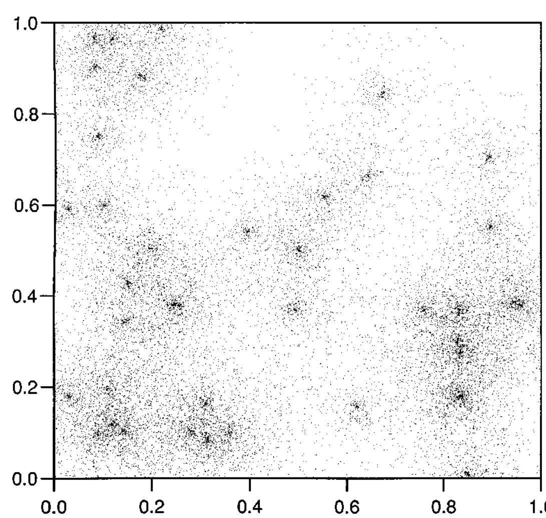

FIG. 3. Distribution of 30 singularities with density profile $n(R) \propto R^{-\alpha}$ with $\alpha = 2$.

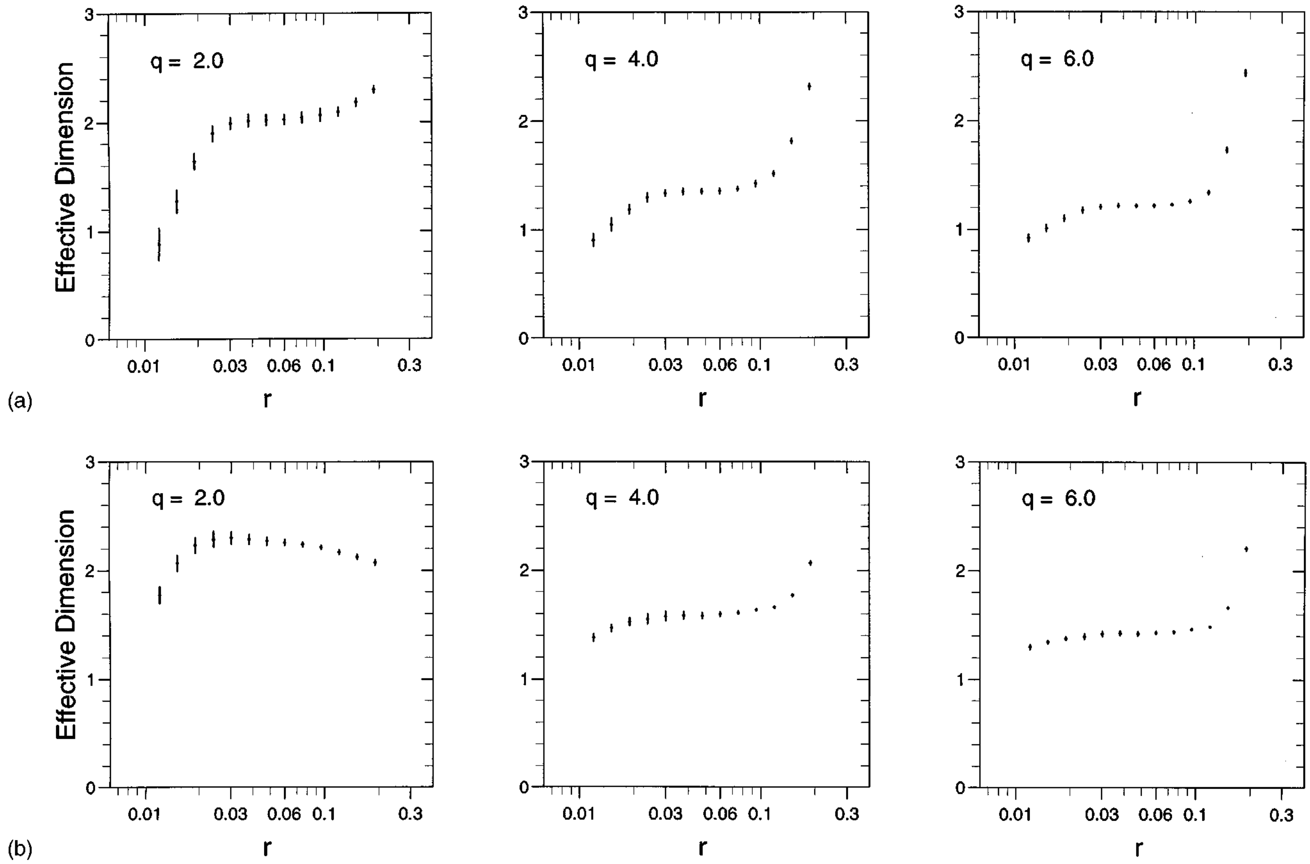

FIG. 4. Results of the scaling analysis of a random distribution of singularities with profile $n(R) \propto R^{-\alpha}$ with $\alpha=2$ (panel a) and $\alpha=1.8$ (panel b). We show $D_q(r)$ for $q=2,4,6$.

In Fig. 3, we show the distribution of points for a random, uncorrelated distribution of 30 singularities with $\alpha=2$. When we calculate the generalized dimensions for the distribution of points in Fig. 3 we get Fig. 4a. This is for $\alpha=2.0$ while Fig. 4b shows the dimensions for a similar random distribution with $\alpha=1.8$. There is a qualitative resemblance of these results to those for the galaxy distribution. The scaling is quite good for these distributions when $q\geqslant 3$. These results mimic the real galaxy distribution in having only a finite number of points and the resulting limited resolution is manifest for the smallest values of q and/or r.

V. GALACTIC LACUNARITY

As the number of available data increases, results of the kind we are displaying will become crisper. At present, we are still in the position of groping for a reasonable picture of the galaxy distribution and its causes. In the previous section, we suggested that the distribution on the smallest scales may be simply an array of singular density concentrations. In this view, the scaling exponent of the galaxy distribution, $\gamma=1.8$, is a consequence of the singularity exponent. But what about the larger scales? We suggest that there are several scale ranges of interest, each with its own scaling behavior. If the smallest scales are singularity driven and the largest scales look space-filling, what happens at the intermediate scales on which the pancakes form?

From the analysis of the ZCAT, we have seen that scaling is present for the higher moments at intermediate scales between about 1 and 15 Mpc. Since there is more than one way to produce an interesting set of generalized dimensions, however, we need a deeper look at the galaxy distribution to decide on its ultimate character. We therefore seek not just the D_q, but also the next term in the development in log r suggested by (3). This time, we go farther and write

$$\log C_q = D_q \log\frac{r}{r_0} + \log\Lambda + O\left(\frac{1}{\log(r/r_0)}\right). \tag{4}$$

If we keep only the first two terms, we have

$$C_q(r) = \Lambda r^{D_q}. \tag{5}$$

What is important here is that as long as $\log\Lambda$ is $O(1)$, it need not be a constant and may depend on log r.

For a monofractal, with $D_q \equiv D$, homogeneity is expressed mathematically by the scaling relation $C_q = AC_q(Br)$, where A and B are constants. A generalization to weakly inhomogeneous fractals might be imagined with A and B dependent on q, but we shall proceed with the homogeneous case, in which the scaling relation admits (5) as the general solution with $D=-\log A/\log B$. Mandelbrot[8] referred to Λ as a prefactor and named it lacunarity. For this

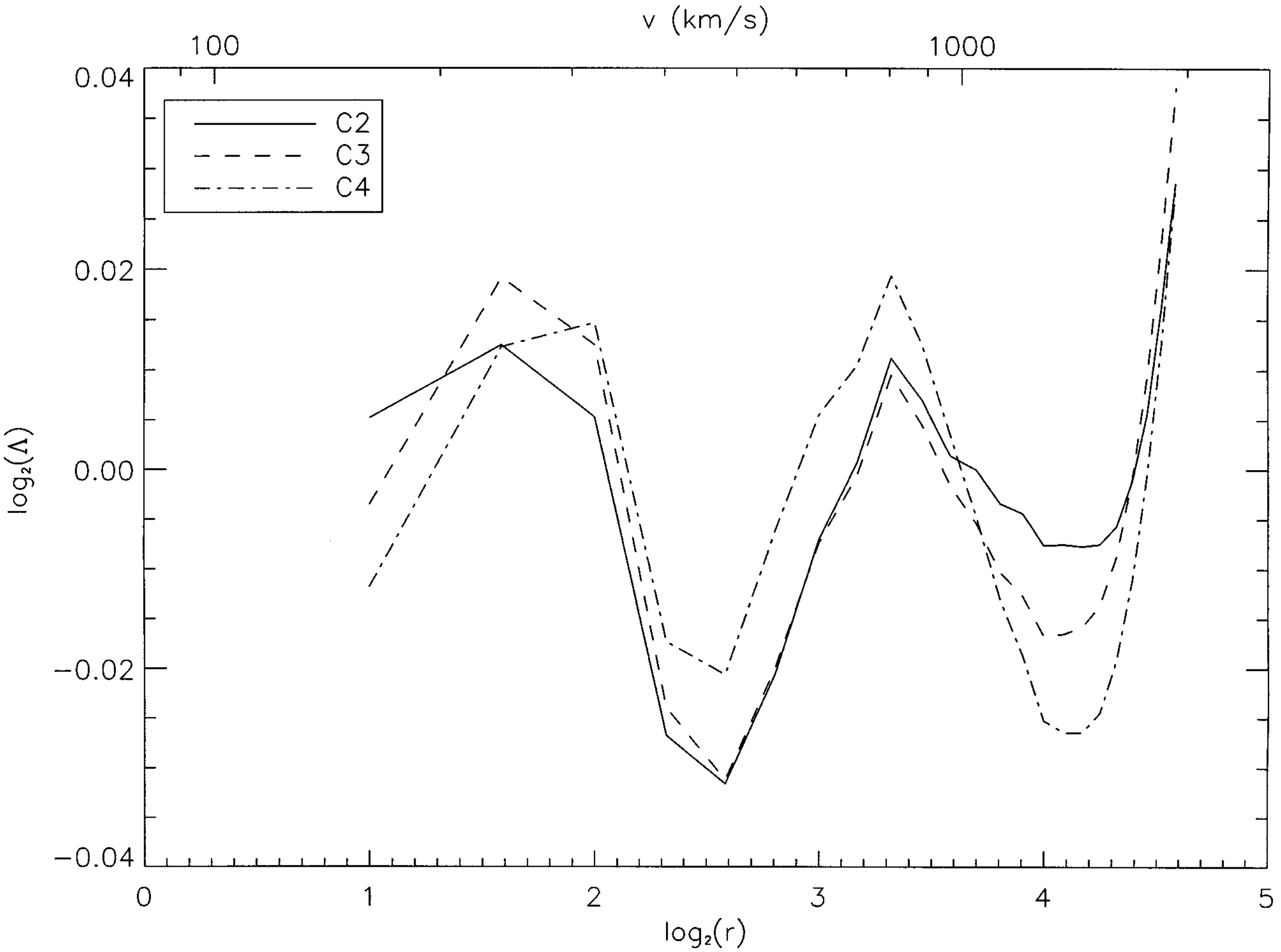

FIG. 5. Log-Log plot of the lacunarity function $\Lambda(r)$ for $q=2$ (solid line), $q=3$ (dashed line) and $q=4$ (dash-dotted line).

more general case, we use the term *lacunarity function* or LF since Λ is a periodic function of $\log(r/r_0)$ with period $P=\log B$. This can be seen by direct substitution of the scaling law into (5). Wherever scaling laws occur, periodic functions like the LF may be expected. Examples are Brownian motion[30] and turbulent dynamics;[31,32] an analogy to Bloch's theorem has recently been drawn by Solis and Tao (unpublished).[33] In the simplest example, Cantor sets on the interval, the lacunarity functions are readily studied in numerical detail[32,34] and are found to be nondifferentiable and complicated. In the case of multifractals, lacunarity functions are less easily found and, if they do appear, we might expect the periods to depend on q. In real life, we might worry whether noise, extraneous to the fractal-making process, would wash out the lacunarity wiggles. In fact, we have found by experimentation that this is not a serious concern except at very high noise levels, and the unpublished investigation of this issue by Solis and Tao supports this finding with a theoretical analysis.

The study of the lacunarity of the galaxy distribution involves first determining the values of D_q as already described and then removing the $D_q \log r/r_0$ term of (4). The remainder is then, to leading order, the LF. To get the LF we resort to a procedure called Extended Self Similarity (ESS), which has been introduced in the study of turbulence.[6] This procedure consists simply in rewriting (3) in the form

$$\log C_{q+1} = \beta_q \log C_q + \dots , \tag{6}$$

where $\beta_q = D_q / D_{q+1}$. This version of (3) permits a somewhat more-error free determination of the higher dimensions. We then calculate the generalized dimensions D_q by a least-square-fit of $\log C_{q+1}$ vs $\log C_q$ over the whole range from 2 to 16 h^{-1} Mpc, once D_2 has been determined using (3). Then, knowing the D_q, we can find Λ using (4).

With this approach, we obtain, for the northern hemisphere of the ZCAT, the results shown in Fig. 5 for the three values of q indicated. All these curves look similar to one another. If we regard them as approximations to periodic functions, their apparent periods show no detectable dependence on q, in support of the supposition of (near) homogeneous scaling. Though the logarithmic amplitudes of the oscillations seen in the figure are modest, they are nevertheless significantly larger than the probable errors we estimate for the D_q. Indeed, as with other fractals, the apparent fluctuations about the relation (3) seen in the data are mainly a manifestation of the LF, as is clear from Fig. 2.

When we began looking for the galactic LF, P. J. E. Peebles informed us that a similar statistical oscillation had already been suggested on different grounds by de Vaucouleurs.[35] His scale and our period are roughly comparable for an r_0 of a few Mpc.

VI. CONCLUSIONS

The distribution of the galaxies is a remnant of earlier conditions in the universe and may reflect the scales of fluctuations from very early times. The problems of quantifying such a distribution provide a wonderful exercise in the analysis of point sets and the theory of (multi-) fractals. As the data become increasingly abundant, we shall be able to go ever more deeply into this subject. We may hope then to understand the process that produces clusters and to understand, for example, the analogy between the cascades that operated to form galaxies, clusters of galaxies, and so on, and the energy spectrum of turbulence. In both cases, the questions of fractional dimensions and oscillatory behavior of the lacunarity function are of great interest. Though it must be recalled that our results apply strictly in redshift space, in may not be too optimistic to believe that the present results reveal the LF for the galaxies. In that case, we may be justified in modeling the formation of structures in the galaxy distribution in terms of cascade processes.

Even at a more elementary level, the detection of the LF has interesting consequences. For skeptics about the interpretation of redshifts as indicators of distance, it would be hard to explain such geometrically suggestive results if the space we were working in here did not have some geometric significance. Moreover, we have found a preliminary indication that the period of the LF is the same in the northern and southern galactic hemispheres. We have not gone into this here as our interest has been more in the fact of the detection of cosmic lacunarity than in its quantitative aspects, and the LF is less well determined in the southern case. But if the agreement on periods holds up, we have a remarkable way to check the notion that the local universe is globally isotropic, even though the local structure is full of voids and local inhomogeneities of various scales.

ACKNOWLEDGMENTS

We are grateful for the help of several colleagues, especially G. Murante, A. Refregier and S. Goshen.

[1] P. J. E. Peebles, *Principles of Physical Cosmology* (Princeton U. P., Princeton, NJ, 1992).
[2] J. P. Huchra, M. J. Geller, C. M. Clemens, S. P. Tokarz, and A. Michel, *CfA Redshift Catalogue*; 10-3-93 version (1993) available at ftp://cdsarc.u-strasbg.fr/pub/cats/VII/164.
[3] M. J. Geller and J. P. Huchra, in *Large-Scale Motions in the Universe*, edited by V. C. Rubin and G. V. Coyne (Princeton U. P., Princeton, NJ, 1988), p. 3.
[4] V. de Lapparent, M. J. Geller, and J. P. Huchra, Astrophys. J. **302**, L1 (1986).
[5] C. Beck and R. Schlögl, *Thermodynamics of Chaotic Systems* (Cambridge U. P., Cambridge, 1986).
[6] R. Benzi, S. Ciliberto, C. Baudet, and G. R. Chavarria, Physica D **80**, 385 (1995).
[7] G. Paladin and A. Vulpiani, Phys. Rep. **156**, 147 (1987).
[8] B. B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, New York, 1982).
[9] P. J. E. Peebles, *The Large Scale Structure of the Universe* (Princeton U. P., Princeton, NJ, 1980).
[10] M. Livio, ST Sci. Newsletter **13**, 13 (1996).
[11] P. Grassberger and I. Procaccia, Physica D **9**, 189 (1983).
[12] P. H. Coleman and L. Pietronero, Phys. Rep. **213**, 311 (1992).
[13] Ya. B. Zel'dovich, Astron. Astrophys. **5**, 84 (1970).
[14] E. W. Kolb and M. S. Turner, *The Early Universe* (Addison–Wesley, Redwood City, CA, 1990).
[15] G. Efstathiou, S. M. Fall, and G. Hogan, Mon. Not. R. Astron. Soc. **189**, 203 (1979).
[16] M. S. Soneira and P. J. E. Peebles, Astron. J. **83**, 845 (1980).
[17] L. Pietronero, Physica A **144**, 257 (1987).
[18] B. J. T. Jones, V. J. Martinez, E. Saar, and J. Einasto, Astrophys. J. **332**, L1 (1988).
[19] V. Martinez and B. J. T. Jones, Mon. Not. R. Astron. Soc. **242**, 517 (1990).
[20] R. Thieberger, E. A. Spiegel, and L. A. Smith, in *The Ubiquity of Chaos*, edited by S. Krasner (Am. Assn. Adv. Sci., 1990), Chap. 16.
[21] A. Provenzale, in *Applying Fractals in Astronomy*, edited by A. Heck and J. M. Perdang (Springer-Verlag, Berlin, 1991), pp. 97.
[22] V. J. Martinez, in *Applying Fractals in Astronomy*, edited by A. Heck and J. M. Perdang (Springer-Verlag, Berlin, 1991), p. 135.
[23] A. Provenzale, P. Galeotti, G. Murante, and B. Villone, Astrophys. J. **401**, 455 (1992).
[24] S. D. M. White, in *Gravitational Dynamics*, edited by O. Lahav, E. Terlevich, and R. Terlevich (Cambridge U. P., Cambridge, 1996).
[25] S. M. Cole and C. G. Lacey, Mon. Not. R. Astron. Soc. (in press); preprint astro-ph/9510047.
[26] R. K. Sheth and B. Jain, submitted to Mon. Not. R. Astron. Soc.; preprint astro-ph/9602103.
[27] J. F. Navarro, C. S. Frenk, and S. D. M. White, Mon. Not. R. Astron. Soc. **275**, 720 (1995).
[28] G. Efstathiou, C. S. Frenk, S. D. M. White, and M. Davis, Mon. Not. R. Astron. Soc. **235**, 715 (1988).
[29] G. Murante, A. Provenzale, E. A. Spiegel, and R. Thieberger, submitted to Mon. Not. R. Astron. Soc.
[30] M. F. Shlesinger, B. D. Hughes, and E. W. Montroll, in *Sixth International Conference Noise in Physical Systems*, edited by P. Meijer, R. Mountain, and R. Soulen (U.S. National Bureau of Standards, Washington, DC, 1982), p. 18.
[31] E. A. Novikov, Sov. Phys. Dokl. **1**, 497 (1966).
[32] L. A. Smith, J.-D. Fournier, and E. A. Spiegel, Phys. Lett. A **114**, 465 (1986).
[33] F. Solis and L. Tao (private communication, unpublished manuscript, 1996).
[34] R. Badii and A. Politii, Phys. Rev. Lett **52**, 1661 (1984).
[35] G. de Vaucouleurs, Science **167**, 1203 (1970).